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A new indicator for financial risk and portfolio investment based on it

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Abstract

Risk management is the key issue in financial investments, while its basis is the indicator for risk. The first risk indicator was suggested in 1950 ' s by Markowitz, which becomes the cornerstone for modern investment theories. Since then various indicators have been proposed for market risk, such as semi-variance, absolute variance, Value-at-Risk (VaR), conditional VaR and many others.

However, all risk indicators for market risk only reflect the risk at a specific point of time; they do not care the risk involved from now and till that future point in time. In real investment situations, most investors may not have a clear idea regarding when they will sell the financial instruments in hand. Investors may know a time span they may hold the financial instruments, but this does not mean that they have to sell the instruments at the end of the time span, they are flexible regarding the termination time of investments, so they care about the risk over the whole investment time span and not just about the risk at the end of the time span. In order to reflect market risk over a period of time, our research group has been advocating a risk indicator named Period Value at Risk (PVaR) in recent years.

This study aims at solving technical issues related to the use of PVaR in investment decision, especially, we explore methods for calculating PVaR of an investment, and methods for solving investment decision models with PVaR included.

To calculate PVaR of an investment, we derive an analytic method for computing PVaR under the conditions that there is only one risk factor which can be modeled as a geometric Brownian motion. For more general cases where multiple risk factors are involved, we propose the use of Monte Carlo simulation to estimate PVaR of an investment. We illustrate this method for a portfolio composed of correlated financial instruments, and the prices of component financial instruments can be expressed with geometric Brownian motions, whose parameters can be identified with historical price data. Our computing experiments show that the simulation method provides a usable way for estimating PVaR, making PVaR operational in investment practice.

To solve investment decision models with PVaR included, we consider portfolio selection problems with PVaR as the indicator of market risk, and PVaR is estimated using Monte Carlo simulation. The risk minimization model turns out to be a complicated nonlinear programming; we suggest solving the model by changing it to an equivalent mixed programming model, and illustrating this method with a numerical example. We do computing experiments with data from financial market, and show that this method is valid for models with a small size, but computing loan increased rapidly when the model size grows.

PVaR is a brand new indicator for market risk over a period of time, this study proposed methods for computing PVaR of an investment and solving investment decision models with PVaR included, these results are expected to facilitate the use of PVaR in making financial investment decisions.

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Contents

1 Introduction **1**

1.1 Introduction 1

1.2 Thesis outline 4

2 A new risk indicator: Period Value at Risk **7**

2.1 Notion of Period Value at Risk 7

2.2 Value-at-Risk 9

2.2.1 Notion of VaR 9

2.2.2 Computation methods of VaR 9

2.2.3 Comparison of PVaR and VaR 10

2.3 Summary 13

3 Estimation of Period Value at Risk: One risk factor case **15**

3.1 Calculation of PVaR 15

3.1.1 An analytic method for calculating PVaR 15

3.1.2 Estimation of PVaR by Monte Carlo simulation 16

3.2 Estimation of PVaR: when the risk factor is a GBM 18

3.2.1 An analytic method 19

3.2.2 Estimation by Monte Carlo Simulation 22

3.3 Estimation of PVaR: when the risk factor is a Jump process . . . 25

3.4 Computation experiments 29

3.5 Summary 33

4 Estimation of Period Value at Risk: Multiple risk factors case 35

4.1 Estimation of PVaR: two risk factors case 37

4.2 Estimation of PVaR: Multiple risk factors case 43

4.3 Computation experiments 46

4.4 Summary 51

5 Portfolio Selection based on Period Value at Risk 53

5.1 Portfolio selection and its formulation 53

5.2 Risk minimization: model and its solution method 55

5.2.1 A solution method 56

5.2.2 Computation experiments 61

5.3 Return Maximization: model and its solution method 63

5.3.1 A model and its solution method 63

5.3.2 Computation experiments 66

5.4 Summary 67

6 Summary and Future Research 69

6.1 Summary 69

6.2 Future Research 70

A Proof of Lemma 1 in Section 3.2 75

B Proof of Proposition 1 in Section 3.2 77

C Correlation Coefficients 79

List of Figures

2.1 Illustration of VaR 11

2.2 Illustration of PVaR 12

3.1 Relationship between PVaR and Confidence Level 22

3.2 Prices of IBM stock 30

3.3 Daily profit rates of IBM stock 30

4.1 The Components of Dow Jones Industrial Average 47

5.1 Set of feasible solutions 60

C.1 Correlation coefficients of 30 stocks 80

C.2 Correlation coefficients of 30 stocks 81

C.3 The historical stock prices of stocks during 2011 82

C.4 The historical stock prices of stocks during 2011 83

C.5 The historical stock prices of stocks during 2011 84

C.6 The historical stock prices of stocks during 2011 85

C.7 The historical stock prices of stocks during 2011 86

C.8 The historical stock prices of stocks during 2011 87

List of Tables

3.1 Estimation of PVaR with Simulation 17

3.2 Computation results from solving Equation(3.13) (μ = 0.3 and σ = 0.2) 22

3.3 The four statistical characteristics of the daily profit rate of IBM stock 31

3.4 Parameters estimated for the GBM model and the jump process model 31

3.5 PVaR and VaR estimated by the numerical method 32

3.6 PVaR of holding the IBM stock for one year 32

4.1 Comparison of the theoretical value and the historical value of variances and covariances 49

4.2 Parameters estimation 50

4.3 PVaR and VaR estimated by the numerical method 51

5.1 Three combinations of five equities 61

5.2 Comparison of the averages 62

5.3 PVaR and the optimal investment decision 62

5.4 Return and the optimal investment decision 67

Chapter 1

Introduction

1.1 Introduction

The measurement of risk is the basis of risk management. Markowitz (1952) [23] was the first to propose a measure of financial risk, he suggested using variance as a measure of financial risk. Variance has been popular since the 1950s, and is the cornerstone of modern investment theories. However, it is not a satisfactory risk measure for two reasons: it penalizes gains and losses in the same way, and it is not connected with loss directly.

Many other risk measures, such as semi-variance, absolute variance (Konno and Yamazaki, 1991)[22], lower partial moments (Bawa and Lindenberg, 1977)[8], Value-at-Risk (VaR, RiskMetricsGroup, 1996)[17], conditional Value-at-Risk (CVaR, Rockafellar and Uryasev, 2000)[29] and maximum loss (Young, 1998)[37], have been proposed to effectively measure financial risk. While most of these risk measures were developed to measure the downside risk, some were developed to improve computational efficiency in solving investment models with risk included. See Dowd (2002)[13] for a survey of market risk measurements.

Value-at-Risk (RiskMetricsGroup, 1996)[17], a well-known risk measure proposed by J.P.morgan, has been popular for over two decades. It was supported by many financial institutes including the Bank for International settlements. It has become a new standard measure in financial industry for its conceptual simplicity and practical application (Basak and Shapiro, 2001, Gaivoronshi and Pflug, 2005)[7, 15].

VaR of a portfolio is the maximum loss we might expect over a given holding or horizon period, at a given level of confidence (Dowd, 2002)[13]. However, VaR is not a coherent risk measure (Artzner, etc, 1997, 1999)[4, 5], and does not indicate the size of the potential loss exceeds VaR. To remedy these shortcomings, several variations of VaR were developed, such as the tail conditional expectation (TCE, Artzner, etc, 1999)[5], worst conditional expectation (WCE, Benati, 2004)[10], expected shortfall (ES, Acerbi, etc, 2001, 2002; Tasche, 2002 and Yamai, etc, 2002)[1, 2, 33, 36], conditional Value-at-Risk (CVaR, Rockafellar and Uryasev, 2000)[29] and entropic value-at-risk (EVaR, Ahmadi-Javid, 2012)[20]. See Duffie and Pan (1977)[14], Danielsson, etc (2013)[12], Holton (2003)[19], Szego (2002)[32] for detailed descriptions on VaR.

TCE is also known as conditional tail expectation (CTE, Brazauskas etc, 2008)[11], or tail value at risk (TVaR). And ES is also called conditional Value-at-Risk (CVaR), average value at risk (AVaR) and expected tail loss (ETL). The difference of WCE, TCE and CVaR is given in (Benati, 2003)[9], a thorough discussion on the theoretical properties of WCE is given in (Artzner etc, 1999, Benati, 2004) [5, 10]

But all these measures have reflected only the risk at certain future point in time, they do not consider the risk involved from now and till that future point in time. In estimating market risk measured by any conventional risk indicators, risk factors are regarded as stochastic variables, such as taking the price of a certain stock at some future time as a stochastic variable, market risk is then measured by the stochastic features of the risk factors. Risk estimated in this way is only the risk at some future point in time, it does not reflect the risk over a period of time in the future.

In real investment situations, most investors may not have a clear idea regarding when they will sell the financial instruments in hand, they may know a time span they may hold the financial instruments. However, this does not mean that they have to sell the instruments at the end of the time span, they are flexible regarding the termination time of the investment. They care about the risk over the whole investment time span and not just about the risk at the end of the time span. In addition, some investment regulations require fund managers to immediately dispose of any financial instruments when their prices decrease by half; hence, the risk within a time span is an important criterion for investors. However, risk indicators to date fail to consider the risk involved over a period of future time.

This study proposes and formulates the notion of financial risk over a period of time. We extend VaR to Period Value at Risk (PVaR) as an indicator of risk for a given time span, and then suggest a method for estimating PVaR numerically, because calculating PVaR of an investment analytically is often difficult. The proposed method is based on Monte Carlo simulation, it provides a usable method for estimating PVaR of an investment, which makes PVaR usable in investment practice. We also attempt to establish models for portfolio selection problems with PVaR as the indicator of market risk, and proposes resolution methods for them.

1.2 Thesis outline

Chapter 2 proposes and formulates the notion of PVaR. This chapter proposes and formulates the notion of Period Value at Risk (PVaR) for reflecting the market risk of an investment over a period of time.

Chapter 3 proposes two methods for calculating PVaR, an analytic method and a Monte Carlo simulation method, while there is only one risk factor. We calculate PVaR for investments when the risk factor can be modeled as a geometric Brownian motion or as a jump process.

We develop an analytic expression for PVaR on the assumption that the risk factor can be expressed as a geometric Brownian motion.

We verify the validity of the Monte Carlo simulation method, by comparing the results obtained from using the simulation method with those obtained from using the analytic method.

Chapter 4 estimates PVaR of a portfolio.

In this chapter we propose a simulation method for estimating PVaR, for the multiple risk factors case. To estimate the PVaR of a portfolio, we assume that multiple risk factors can be described as a multi-dimensional geometric Brownian motion.

Chapter 5 considers portfolio selection problems with PVaR as the risk indicator.

In this chapter we formulate portfolio selection problems with PVaR as the risk indicator, when PVaR is estimated using Monte Carlo simulation. These models are complicated nonlinear programming models, we suggest to solve these models by changing them to equivalent mixed programming models, and illustrate these methods with numerical examples. We do computing experiments with data from financial market, and show that these methods are valid for models with a small size, but computing time increased rapidly when the model size grows.

Finally, Chapter 6 presents the summary and some suggestions for further research.

Chapter 2

A new risk indicator: Period Value at Risk

This chapter proposes and formulates the notion of Period Value at Risk (PVaR) for reflecting the market risk of an investment over a period of time. We begin in Section 2.1 proposing a new indicator for financial risk, PVaR. To illustrate the difference between PVaR and VaR, we review the definition and computation methods of VaR, and compare PVaR with VaR in Section 2.2. Section 2.3 is a summary of this chapter.

2.1 Notion of Period Value at Risk

Market risk is caused by uncertainty of risk factors in future market. Without losing generality, we explain and formulate the notion of Period Value at Risk (PVaR) by focusing on a simple situation in which there is only one risk factor in investments.

Let $\omega(t)$ be the value of the risk factor at time t, which may be regarded as the price of the risk factor. We suppose that $\omega(t) : t \in [0, T]$ is a stochastic process within $[0, T]$ that expresses the uncertainty in future market. Let $L(\omega(t))$ be the loss rate of an investment, which can be determined by the values of $\omega(t)$.

Any indicator for risk should be related directly or indirectly to the loss of an investment. To express the risk over a period of time, we define the largest loss rate of an investment over time span $[0, T]$ as follows,

$$L_T = max\{L(\omega(t)) : t \in [0, T]\}. \tag{2.1}$$

Obviously, L_T is a stochastic variable. The largest loss rate that may occur at probability α, denoted by d^{min}, is given by

$$d^{min} = inf\{d \in R^1 | P(L_T > d) \leq \alpha\}. \tag{2.2}$$

We define such a value d^{min} as the Period Value at Risk at confidence level $1-\alpha$, denoted by $PVaR_{1-\alpha}$. Because $PVaR_{1-\alpha}$ is the largest loss rate that may occur at probability α, we take it as an indicator of the risk over time period $[0, T]$.

Definition 1 *The Period Value at Risk (*PVaR*) at confidence level* $1 - \alpha$ *of an investment is defined in the following formula:*

$$PVaR_{1-\alpha} = inf\{d \in R^1 | P(L_T > d) \leq \alpha\}, \tag{2.3}$$

which is the largest loss rate that may occur with a certain probability within a period of time.

Remark 1 *Taking $\omega(t)$ in formula (2.1) as a vector of risk factors, we can see that the above definition of PVaR applies to cases in which there are many risk factors; to save space, we do not rewrite the formulation.*

Formula (2.3) gives only the definition of PVaR of an investment; how to calculate the PVaR of an investment is a challenging issue. We will discuss it in next chapter.

2.2 Value-at-Risk

To see the difference between PVaR and conventional VaR, we review the notion of the VaR.

2.2.1 Notion of VaR

Value-at-risk (Yamai and Yoshiba, 2002)[36] , in general, is defined as " the possible maximum loss over a given holding period within a fixed confidence level". The mathematical definition is as follows.

Let $\alpha\epsilon(0, 1)$ be a given probability level and L is the loss of an investment, VaR at confidence level $1 - \alpha$ of an investment is defined in the following formula:

$$VaR_{1-\alpha} = inf\{d \in R^1 | P(L > d) \leq \alpha\}, \tag{2.4}$$

For example, $VaR_{0.95} = 100$ means that the probability of the loss being greater than 100 is not more than 5%.

2.2.2 Computation methods of VaR

The methods for calculating VaR can be classified into the following three groups (Alexander, 2009)[3], (Pritsker, 1997)[27].

(1)The normal method

The most basic assumption of this method is that the profit rates of risk factors are normally distributed, and that their joint distribution is multivariate normal; therefore, only the covariance matrix of risk factor returns is required to capture the dependency between the risk factors.

(2)The historical method

The historical method assumes that all future variations occur with the same probability as in the past. A distribution of loss can be obtained using the historical data.

(3)The simulation or Monte Carlo simulation method

The simulation method requires a distribution of each risk factor including correlations between factors. Monte Carlo simulation is used to obtain simulated changes in the risk factors, which are used to obtain a profit/loss distribution in the same manner as in the historical method.

2.2.3 Comparison of PVaR and VaR

VaR reflects only the risk at a certain point in time T, while PVaR is related to the risk over a time span $[0, T]$. From the Figure 2.1 and Figure 2.2 we can see that there is a significant difference between PVaR and VaR. PVaR reflects the largest loss within a period of time. VaR reflects the maximum of loss at the end of a time period.

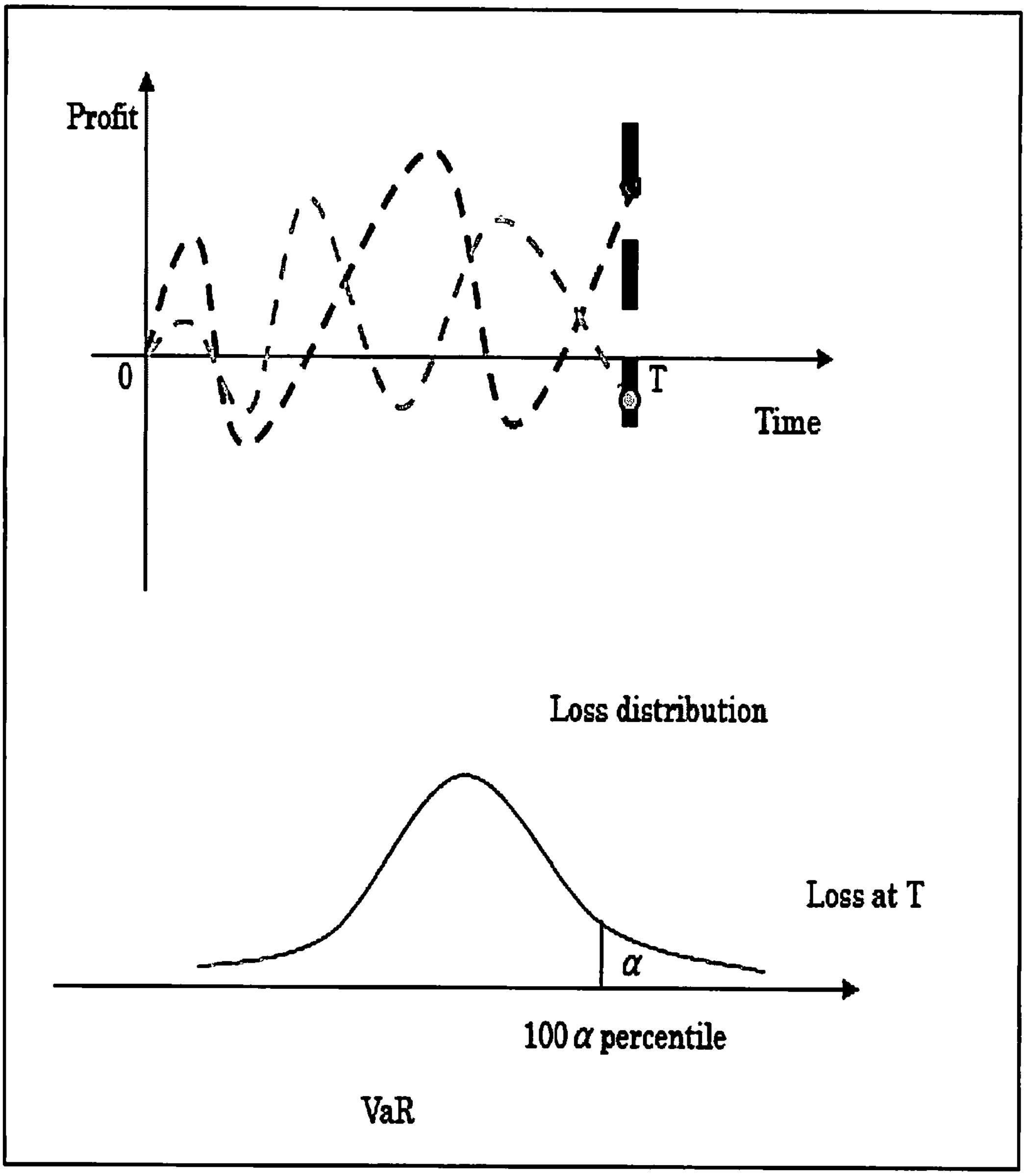

Figure 2.1: Illustration of VaR

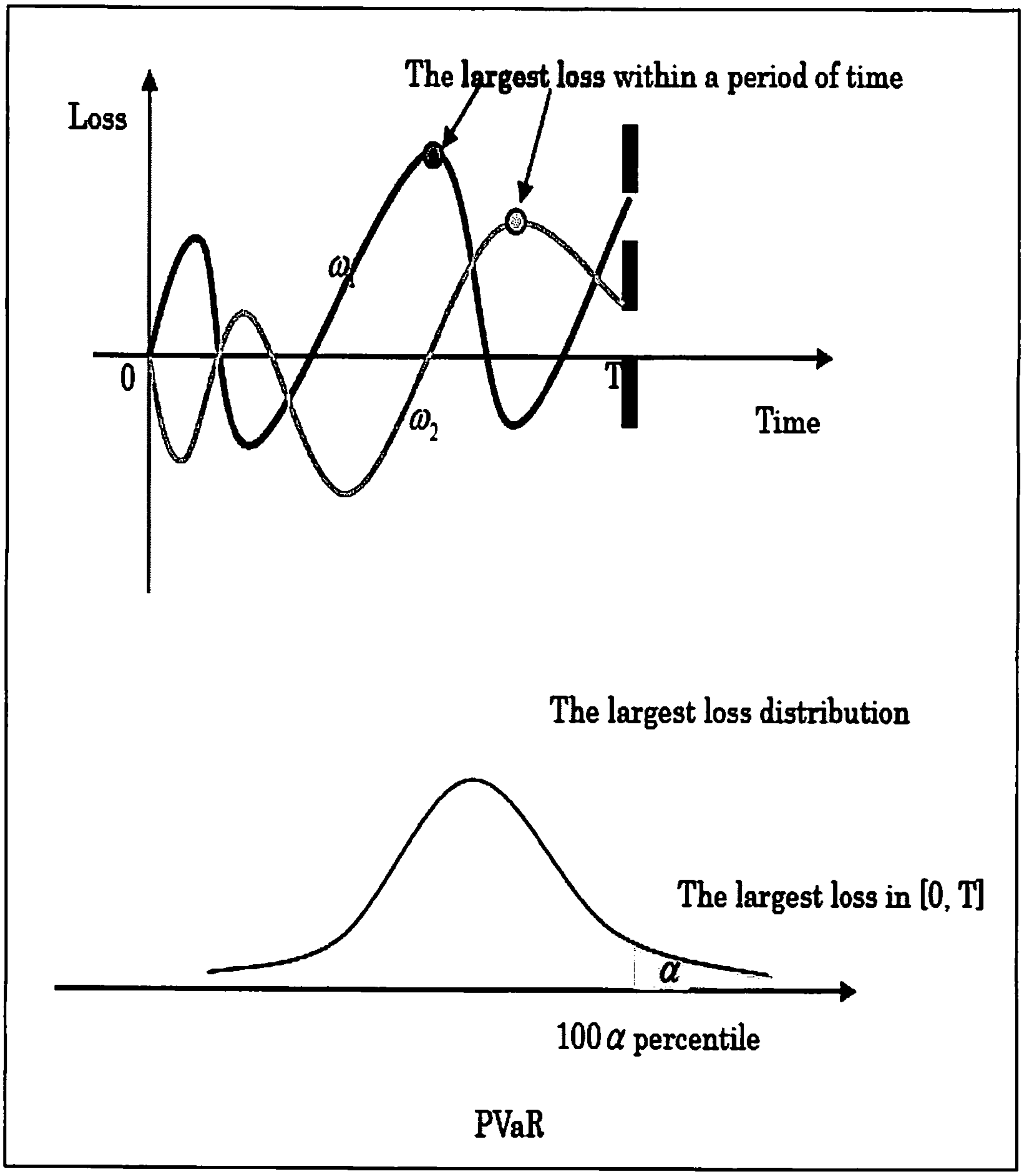

Figure 2.2: Illustration of PVaR

2.3 Summary

This chapter proposed and formulated the notion of Period Value at Risk (PVaR) for reflecting the market risk of an investment over a period of time. We showed that PVaR is different with VaR, PVaR reflects the largest loss within a period of time, while VaR reflects the maximum of loss at the end of a time period.

Chapter 3

Estimation of Period Value at Risk: One risk factor case

In this chapter we propose two methods to calculate PVaR, for the case that there is only one risk factor. Section 3.1 considers the computation method for PVaR by focusing on a simple case in which there is one risk factor in investments. Section 3.2 estimates PVaR under the geometric Brownian motion assumption. Section 3.3 estimates PVaR under the jump process assumption. These methods will be illustrated and compared in Section 3.4. Section 3.5 is a summary of this chapter.

3.1 Calculation of PVaR

3.1.1 An analytic method for calculating PVaR

From the definition of PVaR we can know that the cumulative distribution function of the largest loss, L_T, is very important during the calculation. When we know the cumulative distribution function of the largest loss in period [0, T], denote it by $F(L_T)$, then PVaR at confidence level $1-\alpha$ is $F^{-1}_{1-\alpha}(L_T)$.

However, obtaining a cumulative distribution function of the largest loss over a period of time for a portfolio is not easy. Hence, getting an analytical solution of PVaR is likely to be difficult when the value of risk factor is a general stochastic process, this study will propose a method for estimating PVaR numerically when the value of the risk factor can be simulated with Monte Carlo simulation.

3.1.2 Estimation of PVaR by Monte Carlo simulation

Suppose we can generate scenarios for the stochastic process $\omega(t) : t \in [0, T]$, each scenario consists of the values of the stochastic process at D different time spots, $t_1, \cdots, t_D$. Let $\omega^k(t_d)(d = 1, 2, \cdots, D, t_d \in [0, T])$ be values of the stochastic process at the k^{th} scenario.

Suppose the loss rate of an investment in each scenario can be calculated. In that case, let the loss rate in the k^{th} scenario at time t_d be $L(\omega^k(t_d))$; then the largest loss rate in the k^{th} scenario, denoted by L_T^k, is given by

$$L_T^k = max\{L(\omega^k(t_d)) : d = 1, 2, \cdots, D\}. \tag{3.1}$$

Let N be the number of simulated stochastic processes, and denote the K^{th} smallest value in set $\{L_T^k, k = 1, 2, \cdots, N\}$ by $L_T^{(K)}$.

If K is set to be the smallest integer among the integers larger than $(1 - \alpha)N$, such a value is to be denoted by $\lceil(1 - \alpha)N\rceil$; then the proportion of the loss rates greater than $L_T^{(K)}$ in $\{L_T^k, k = 1, 2, \cdots, N\}$ will not be greater than α. According to the definition of PVaR, $L_T^{(K)}$ is PVaR at confidence level $1 - \alpha$, i.e.,

$$PVaR_{1-\alpha} = L_T^{(\lceil(1-\alpha)N\rceil)}. \tag{3.2}$$

To clearly illustrate the above method, we enter the data of the loss rate into a table, with each row filled with loss rates in each scenario and the rightmost column filled with the largest data in each row, as shown in Table 3.1. Then PVaR at confidence level $1-\alpha$ is the $\lceil(1-\alpha)N\rceil^{th}$ smallest value in the rightmost column.

Table 3.1: Estimation of PVaR with Simulation

	$L(\omega^k(t_1))$	$L(\omega^k(t_2))$	$\cdots$	$L(\omega^k(t_D))$	$L_T^k = max\{L(\omega^k(t_d)) : d = 1 \sim D\}$
ω^1			$\cdots$		
ω^2			$\cdots$		
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
ω^N			$\cdots$		

Hence when the value of risk factor can be simulated we can estimate PVaR numerically as stated above. Monte Carlo simulation is a kind of simulation method widely used in financial engineering, it produces data for stochastic variables which follow certain kind of probability distributions. Refer to Glasserman (2003)[16] and Wang (2012)[34] for details about Monte Carlo simulation and its applications in finance.

In order to simulate the value of the risk factor using Monte Carlo simulation, it is generally necessary to make certain assumption about the risk factor. When the risk factor in an investment is the price of certain stock, there are several popular assumptions we can adopt.

One is the geometric Brownian motion assumption. The Brownian motion was introduced to finance by Bachelier (1900)[6], Samuelson (1965,1973)[30, 31] presented the argument that the geometric Brownian motion is a good model for stock prices. The application of stochastic calculus to finance began with the work of Merton (1969) [24]. The famous Black-Scholes-Merton formula is based on the geometric Brownian motion model for stock prices.

Another is the jump process assumption, which was reported to be more suitable in modeling stock prices. The easiest place to read about stochastic calculus for processes with jumps is Protter (2004)[28].

It will be discussed in the following sections.

3.2 Estimation of PVaR: when the risk factor is a GBM

We consider a case in which we can make a certain type of assumption about the risk factor. When the risk factor is the price of an equity, one popular assumption about equity price is the geometric Brownian motion assumption; that is,

Assumption 1 *The value of the risk factor can be modeled as a geometric Brownian motion. That is, the value of the risk factor, denoted by $S(t)$, is the solution of the following equation,*

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t), \tag{3.3}$$

where μ, σ are parameters describing a geometric Brownian motion, and $B(t)$ is the standard Brownian motion.

Refer to Karatzas and Shreve(1991) [21] for an introduction to Brownian motion. We derive an analytic formula to calculate PVaR under this assumption.

3.2.1 An analytic method

Under assumption 1, when the purchase price in $S(0)$, the stock price can be expressed by the following formula(Nishida, 2005) [26]:

$$S(t) = S(0)exp[(\mu - \sigma^2/2)t + \sigma B(t)], \quad (3.4)$$

where $S(t)$ is the future value of the price at time t, while μ, σ are two parameters, and $B(t)$ is a standard Brownian motion.

Define the profit rate of such an investment, denoted by $R(t)$, as the logarithmic yield, i.e.,

$$R(t) = ln\frac{S(t)}{S(0)}, \quad (3.5)$$

then we can easily show that the profit rate of such an investment is given by

$$R(t) = (\mu - \sigma^2/2)t + \sigma B(t). \quad (3.6)$$

Define loss rate as the negative profit rate. Then, the loss rate of an investment is given by

$$L(t) = -(\mu - \sigma^2/2)t - \sigma B(t). \quad (3.7)$$

The loss $L(t)$ is a Brownian motion with drift, in formula (3.7), we will make use of the properties of Brownian motion to calculate PVaR. We first state two conclusions related to Brownian motion with drift as follows.

Lemma 1 *Let $X(t)$ be a linear Brownian motion with drift: $X(t) = at+B(t)$, where $B(t)$ is the standard Brownian motion and a is a constant parameter. Denote the maximum value of $X(t)$ within $[0,T]$ by X_T: $X_T = max_{0\le t\le T}X(t)$. Then the joint distribution function of X_T and $X(T)$ is given as follows:*

$$G(x,y) = P(X_T \le x, X(T) \le y) = \Phi(\frac{y-aT}{\sqrt{T}}) - exp(2ax)\Phi(\frac{y-2x-aT}{\sqrt{T}}), \quad (3.8)$$

where $\Phi(\cdot)$ is the cumulative distribution function of the normal distribution.

See Appendix A for a proof of Lemma 1.

Define another linear Brownian motion with drift as follows:

$$Y(t) = at + \sigma B(t). \tag{3.9}$$

Let $Y_T = max_{0 \le t \le T} Y(t)$; then the joint distribution function of Y_T and $Y(T)$ can be calculated as follows:

$$F(x,y) = P(Y_T \le x, Y(T) \le y) = P(\frac{Y_T}{\sigma} \le \frac{x}{\sigma}, \frac{Y(T)}{\sigma} \le \frac{y}{\sigma}).$$

Because $\frac{Y(t)}{\sigma} = \frac{a}{\sigma}t + B(t)$ holds, Lemma 1 implies that the joint distribution function of Y_T and $Y(T)$ is given as follows:

$$\begin{aligned} F(x,y) &= \Phi(\frac{\frac{y}{\sigma} - \frac{a}{\sigma}T}{\sqrt{T}}) - exp(\frac{2a}{\sigma}\frac{x}{\sigma})\Phi(\frac{\frac{y}{\sigma} - 2\frac{x}{\sigma} - \frac{a}{\sigma}T}{\sqrt{T}}) \\ &= \Phi(\frac{y - aT}{\sigma\sqrt{T}}) - exp(\frac{2ax}{\sigma^2})\Phi(\frac{y - 2x - aT}{\sigma\sqrt{T}}). \end{aligned}$$

Hence, we have the following conclusion.

Lemma 2 *Let $Y(t) = at + \sigma B(t)$ be a linear Brownian motion with drift, and $Y_T = max_{0 \le t \le T} Y(t)$; then the joint distribution function of Y_T and $Y(T)$ is given as follows:*

$$F(x,y) = P(Y_T \le x, Y(T) \le y) = \Phi(\frac{y - aT}{\sigma\sqrt{T}}) - exp(\frac{2ax}{\sigma^2})\Phi(\frac{y - 2x - aT}{\sigma\sqrt{T}}). \tag{3.10}$$

The loss rate given in (3.7) is a linear Brownian motion with drift. We can get the cumulative distribution function of L_T, the largest loss rate in period $[0, T]$, as stated in the following proposition.

Proposition 1 *When the value of a risk factor can be modeled as a geometric Brownian Motion, as stated in Assumption 1, the cumulative distribution function of the largest loss rate in period* $[0, T]$*, denoted by* $F(x)$*, is given by*

$$\begin{aligned} F(x) = \ & P(L_T \leq x) = \Phi(\tfrac{x+(\mu-\sigma^2/2)T}{\sigma\sqrt{T}}) \\ & -exp(\tfrac{-2x(\mu-\sigma^2/2)}{\sigma^2})\Phi(\tfrac{-x+(\mu-\sigma^2/2)T}{\sigma\sqrt{T}}). \end{aligned} \tag{3.11}$$

See Appendix B for a proof of Proposition 1.

Let the inverse function of F be F^{-1}. We know from Definition 1 that PVaR at confidence level $1-\alpha$ is $F^{-1}(1-\alpha)$, i.e.,

$$PVaR_{1-\alpha} = F^{-1}(1-\alpha). \tag{3.12}$$

Therefore, $PVaR_{1-\alpha}$ is the solution of the following equation with x as the variable:

$$\Phi(\frac{x+(\mu-\sigma^2/2)T}{\sigma\sqrt{T}})-exp(\frac{-2x(\mu-\sigma^2/2)}{\sigma^2})\Phi(\frac{-x+(\mu-\sigma^2/2)T}{\sigma\sqrt{T}}) = 1-\alpha. \tag{3.13}$$

Solving equation (3.13) analytically is hard; however, we can numerically determine the solution by using common commercial software. An example of such software is MATLAB, a numerical computing environment developed by MathWorks.

We employ MATLAB in solving equation (3.13) for different confidence levels by setting $\mu = 0.3$ and $\sigma = 0.2$. The results are summarized in Table 3.2 below.

Table 3.2 is also plotted in Figure 3.1 below. As anticipated, PVaR progressively increases with confidence level.

Table 3.2: Computation results from solving Equation(3.13) ($\mu = 0.3$ and $\sigma = 0.2$)

$1-\alpha$	0.80	0.82	0.85	0.87	0.90	0.92	0.95	0.97	0.99
PVaR	0.1092	0.1161	0.1280	0.1373	0.1541	0.1683	0.1976	0.2286	0.2926

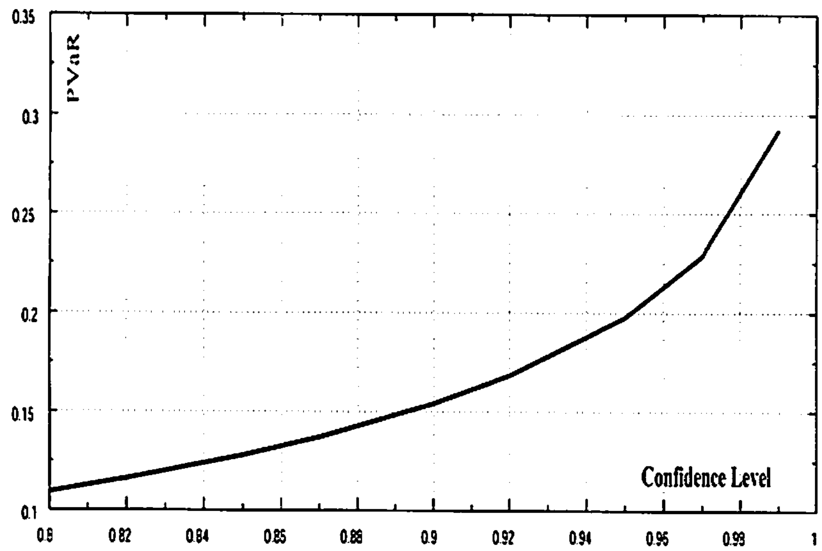

Figure 3.1: Relationship between PVaR and Confidence Level

3.2.2 Estimation by Monte Carlo Simulation

When the risk factor in an investment is stock price, we assume that the risk factor can be modeled as a geometric Brownian motion.

Under Assumption 1, the stock price can be expressed with the following formula:

$$S(t) = S(0)exp[(\mu - \sigma^2/2)t + \sigma B(t)], \tag{3.14}$$

where $S(0)$ is current price of the stock, and $S(t)$ is the price at time t, while μ, σ are some parameters, and $B(t)$ is the standard Brownian motion.

To simulate the stock price with formula (3.14), it is necessary to know the parameters in the formula. We here estimate these parameters using the historical price data of the stock.

Suppose we have $D + 1$ sequential close daily prices of the stock in past T_1 years, denoted by $S(t_0^1)$, $S(t_1^1)$, $S(t_2^1)$, $\cdots$, $S(t_D^1)$. The elapsed time between any two sequential time spots is $\Delta t = T_1/D$. The rate of profit in two adjacent times, measured by the logarithm yield, is

$$r_1(t_d^1) = ln\frac{S(t_d^1)}{S(t_{d-1}^1)}, d = 1, 2, \cdots, D. \tag{3.15}$$

Denote the expectation and variance of data $r_1(t_1^1), r_1(t_2^1), \cdots, r_1(t_D^1)$ by E_1, V_1, respectively, then

$$E_1 = \frac{1}{D}\sum_{d=1}^{D} r_1(t_d^1), \quad V_1 = \frac{1}{D-1}\sum_{d=1}^{D}(r_1(t_d^1) - E_1)^2. \tag{3.16}$$

Denote two adjacent points of time in $[0, T]$ with time interval Δt by t_{d-1} and t_d, and the profit rate in the two adjacent times measured by the logarithm yield by $r(t_d)$. Because the stock price is supposed to be a geometric Brownian motion as described by (3.14), we have the following relationship:

$$r(t_d) = ln\frac{S(t_d)}{S(t_{d-1})} = (\mu - \sigma^2/2)\Delta t + \sigma(B(t_d) - B(t_{d-1})). \tag{3.17}$$

Denote the expectation and variance of $r(t_d)$ defined in (3.17) by $E(r(t_d))$ and $V(r(t_d))$, respectively. Because $B(t)$ is a standard Brownian motion, according to the properties of standard Brownian motion, we know that $B(t_d) - B(t_{d-1}) \sim N(0, \Delta t)$ holds. Hence, we have the following result:

$$E(r(t_d)) = (\mu - \sigma^2/2)\Delta t, \quad V(r(t_d)) = \sigma^2\Delta t. \tag{3.18}$$

Obviously, the expectation and variance of $r(t_d)$ rely on parameters μ and σ. One way to determine μ and σ is to make (3.16) consistent with (3.18), which will make the expectation and variance of the profit rate in the two adjacent times derived from Assumption 1 consistent with those obtained by using historical data. Hence, we determine parameters μ and σ by solving the following equations:

$$\begin{cases} (\mu - \sigma^2/2)\Delta t &= E_1 \\ \sigma^2 \Delta t &= V_1 \end{cases} \tag{3.19}$$

which results in

$$\mu = \frac{E_1 + V_1/2}{\Delta t}, \quad \sigma = \sqrt{\frac{V_1}{\Delta t}}. \tag{3.20}$$

To simulate the stock price described with formula (3.14) wherein the standard BM is included, we first simulate $R(t) = ln\frac{S(t)}{S(0)}$. If we can simulate $R(t)$, then stock prices can be simulated using the following relation:

$$S(t) = S(0)e^{R(t)}. \tag{3.21}$$

According to formula (3.14), we have,

$$R(t) = (\mu - \sigma^2/2)t + \sigma B(t). \tag{3.22}$$

Since $B(t) \sim N(0, t)$ holds, the following expression is an equivalent formula for $R(t)$,

$$R(t) = (\mu - \sigma^2/2)t + \sigma\sqrt{t}\xi, \tag{3.23}$$

where ξ is a variable following the standard normal distribution.

Because many software are available for simulating the standard normal distribution, we can easily simulate $R(t)$ using (3.23), then simulate $S(t)$ using (3.21).

After producing the stock price in each scenario, we then calculate the loss rate in each simulated scenario using the following formula,

$$L(\omega^k(t_d)) = -\frac{S^k(t_d) - S(0)}{S(0)}, d = 1, 2, \cdots, D; k = 1, 2, \cdots, N. \tag{3.24}$$

where $S^k(t_d)$ is the stock price at time t_d on the k^{th} simulated scenario.

Hence we can estimate PVaR under Assumption 1 in the following steps:

Step 1. Compute profit rate data from stock price data in the past using (3.15)

Step 2. Calculate the expectation and variance of profit rates using (3.16)

Step 3. Estimate parameters in formula (3.14) using (3.20)

Step 4. Simulate $R(t)$ using (3.23), and $S(t)$ using (3.21)

Step 5. Compute loss rates in each simulated scenario using (3.24)

Step 6. Estimate PVaR using formulas (3.1) and (3.2)

3.3 Estimation of PVaR: when the risk factor is a Jump process

Assumption 2 *The risk factor in an investment is stock price, which can be modeled as a jump process[25].*

We use the model that was first proposed in Merton(1976) [25], where one plus the jump size has a lognormal distribution.

Let $N(t)$ be a Poisson process with intensity λ, and $Y_1, Y_2, \cdots$ be a sequence of lognormal distributed random variables with mean μ_Y and variance σ_Y^2. The random variables $Y_1, Y_2, \cdots$ are mutually independent and also independent of the Poisson process $N(t)$.

Let $\beta = E(Y_{i1} - 1) = exp(\mu_Y + \sigma_Y^2/2) - 1$, then the stock price can be expressed by the following formula:

$$S(t) = S(0)exp[(\mu - \beta\lambda - \sigma^2/2)t + \sigma B(t)] \prod_{i1=1}^{N(t)} Y_{i1}, \tag{3.25}$$

where $S(0)$ is current price of the stock, $S(t)$ is the stock price at time t; μ, σ are parameters, and $B(t)$ is a standard Brownian motion.

To simulate the stock price with formula (3.25), it is necessary to determine the five parameters $(\mu, \sigma, \lambda, \mu_Y, \sigma_Y)$ in the formula. We estimate these parameters using the historical price data of the stock, similarly as in the previous section.

Suppose we have $D + 1$ sequential close daily prices of the stock in past T_2 years, denoted by $S(t_0^2)$, $S(t_1^2)$, $S(t_2^2)$, $\cdots$, $S(t_D^2)$. The elapsed time between any two sequential time spots is $\Delta t = T_2/D$. The rate of profit in two adjacent times, measured by the logarithm yield, is

$$r_2(t_d^2) = ln\frac{S(t_d^2)}{S(t_{d-1}^2)}, d = 1, 2, \cdots, D. \tag{3.26}$$

We can easily compute the expectation, variance, skewness and kurtosis of data $r_2(t_1^2), r_2(t_2^2), \cdots, r_2(t_D^2)$, denote them by M_1, M_2, M_3 and M_4, respectively.

Denote two adjacent points of time in $[0, T]$ with time interval Δt by t_{d-1} and t_d, and the profit rate in the two adjacent times measured by the logarithm yield by $r(t_d)$. Because the stock price is supposed to be a jump process as described by (3.25), we have the following relationship:

$$r(t_d) = ln\frac{S(t_d)}{S(t_{d-1})} = (\mu - \beta\lambda - \sigma^2/2)\Delta t + \sigma(B(t_d) - B(t_{d-1})) + \sum_{i=N(t_{d-1})}^{N(t_d)} lnY_{i1}. \tag{3.27}$$

We can know from Hisata(2003)[18] that the expectation, variance, skewness and kurtosis of the profit rate described by formula (3.27), denoted by $E(r(t_d))$, $V(r(t_d))$, $Skew(r(t_d))$ and $Kurt(r(t_d))$ respectively, are given as follows, respectively.

$$\begin{array}{lcl} E(r(t_d)) & = & ((\mu - \beta\lambda - \sigma^2/2) + \lambda\mu_Y)\Delta t \\ V(r(t_d)) & = & (\sigma^2 + \lambda(\mu_Y^2 + \sigma_Y^2))\Delta t \\ Skew(r(t_d)) & = & \frac{\lambda\Delta t(\mu_Y^3 + 3\mu_Y\sigma_Y^2)}{((\sigma^2+\lambda(\mu_Y^2+\sigma_Y^2))\Delta t)^{\frac{3}{2}}} \\ Kurt(r(t_d)) & = & \frac{\lambda\Delta t(\mu_Y^4+6\mu_Y^2\sigma_Y^2+3\sigma_Y^4)}{((\sigma^2+\lambda(\mu_Y^2+\sigma_Y^2))\Delta t)^2} \end{array} \tag{3.28}$$

Obviously, the above statistic features of $r(t_d)$ rely on the five parameters. Similarly as in previous section, we can determine these parameters through making the statistic features of the profit rate in the two adjacent times derived from Assumption 2 consistent with those obtained by using historical data, that is, we determine the these parameters by solving the following equations:

$$\begin{cases} E(r(t_d)) & = & M_1 \\ V(r(t_d)) & = & M_2 \\ Skew(r(t_d)) & = & M_3 \\ Kurt(r(t_d)) & = & M_4 \end{cases} \tag{3.29}$$

Since the number of variables in the set of equations (3.29) is five but the number of equations is four, the solution of (3.29) is usually not unique, but any solution may be taken as the values of the five parameters. In unusual circumstances, the set of equations (3.29) may not have a solution, then we can determine the parameters by minimizing the difference between the theoretical value and historical value of the four statistic characteristics, that is, we determine the parameters by solving the following minimization model.

$$\min_{(\mu,\sigma,\lambda,\mu_Y,\sigma_Y)} (E(r(t_d))-M_1)^2+(V(r(t_d))-M_2)^2+(Skew(r(t_d))-M_3)^2+(Kurt(r(t_d))-M_4)^2 \tag{3.30}$$

After determining these parameters, we can produce data for the jump process using Monte Carlo simulation.

Similarly as in previous section, we first simulate the profit rate $R(t) = ln\frac{S(t)}{S(0)}$, and then stock prices using relation (3.21).

According to formula (3.25), we have,

$$R(t) = (\mu - \beta\lambda - \sigma^2/2)t + \sigma B(t) + \sum_{i1=1}^{N(t)} lnY_{i1}. \tag{3.31}$$

Since $B(t) \sim N(0, t)$ holds, the following is an equivalent expression for $R(t)$,

$$R(t) = (\mu - \beta\lambda - \sigma^2/2)t + \sigma\sqrt{t}\xi + \sum_{i1=1}^{N(t)} lnY_{i1}, \tag{3.32}$$

where ξ is a variable following the standard normal distribution.

Simulating $R(t)$ given by (3.32) is easy, because many software, say Matlab developed by MathWorks for example, are available for simulating the standard normal distribution, the poisson process and the lognormal distributed random variable.

After producing the stock prices in each scenario, we then calculate the loss rate in each simulated scenario using formula(3.24), and then estimate PVaR using formulas (3.1) and (3.2).

This chapter proposes a numerical method for estimating the PVaR of an investment where only one risk factor is involved, this method makes PVaR, an indicator of risk over a period of time, usable in investment practice.

We explain this method in details where the risk factor can be modeled as a geometric Brownian motion or as a jump process, which are two popular assumptions about stock prices. However, the proposed numerical method for estimating

PVaR does not rely on the two assumptions, it applies to any situations so long as the risk factor can be simulated numerically.

3.4 Computation experiments

In order to illustrate the method proposed in this chapter for estimating PVaR, and also for showing the difference between VaR and PVaR, we estimate the market risk of an investment to IBM stock, in this section.

Suppose that the maximum holding period of IBM stock is one year, and the investment starts on May 1, 2012.

We estimate the market risk under Assumption 1 and Assumption 2, separately. To estimate the parameters in the GBM model (3.14) and the jump process model(3.25), we use the historical price data of IBM stock from 2009/1/2 to 2011/12/30, which was downloaded from Yahoo!Finance.

With the historical price data, the daily profit rate of IBM stock can be calculated easily. The historical stock prices of IBM stock and the daily profit rate during 2009/1/2 and 2011/12/30 are shown in Figure 3.2 and Figure 3.3 respectively.

We calculate the four statistical characteristics of the daily profit rate of IBM stock using data of the daily profit rate during 2009/1/2 and 2011/12/30, which are showed in Table 3.3 below.

Then the two parameters (μ, σ) in GBM model (3.14) can be calculated using formula (3.20). While the five parameters $(\mu, \sigma, \lambda, \mu_Y, \sigma_Y)$ in the jump model (3.25) can be calculated by solving optimization model (3.30). We use Matlab

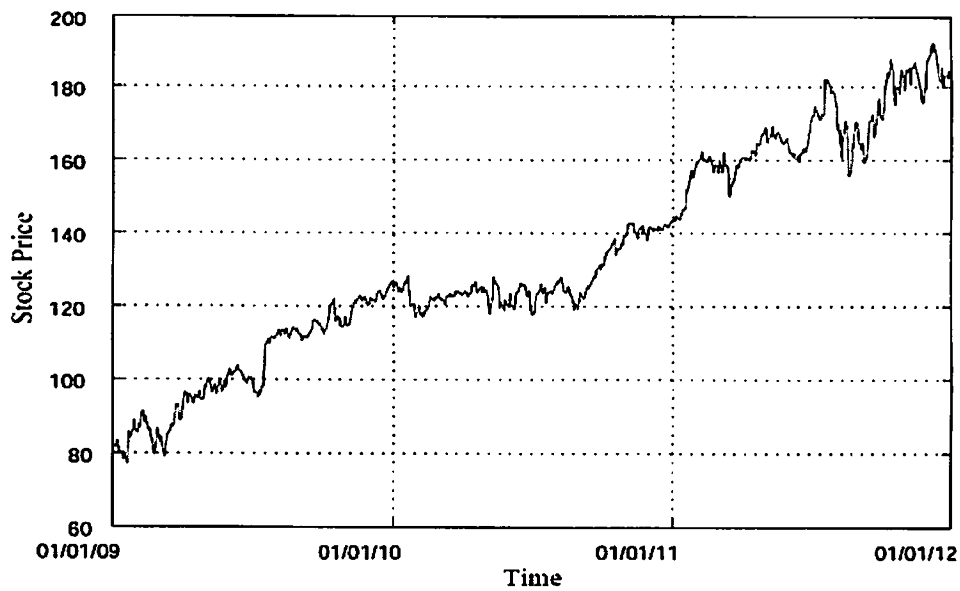

Figure 3.2: Prices of IBM stock

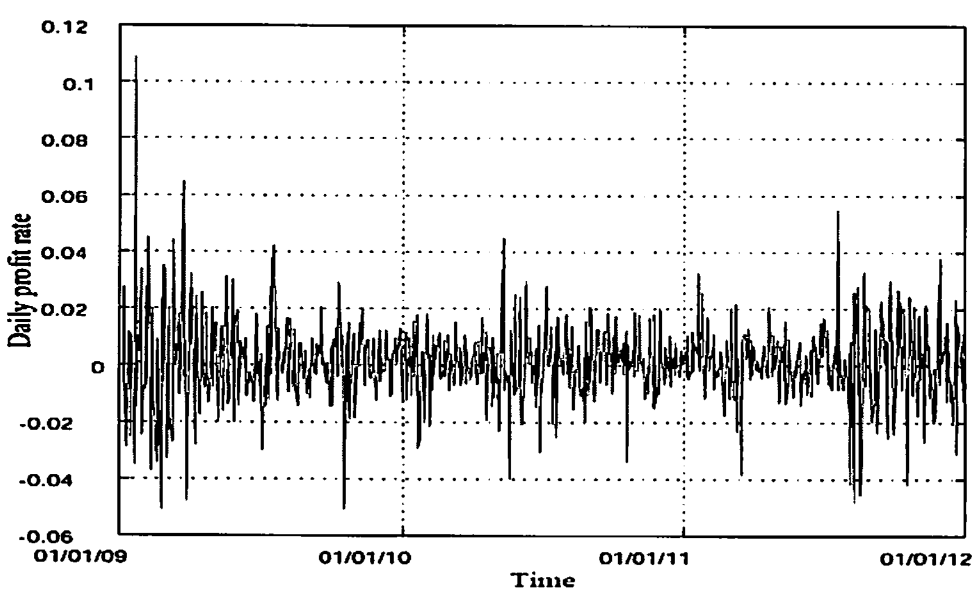

Figure 3.3: Daily profit rates of IBM stock

Table 3.3: The four statistical characteristics of the daily profit rate of IBM stock

Expectation M_1	0.001059
Variance M_2	0.000207
Skewness M_3	0.486523
Kurtosis M_4	5.714537

developed by MathWorks in doing these computation, we omit the details of these computations here, but summarize the results in Table3.4 below.

Table 3.4: Parameters estimated for the GBM model and the jump process model

Parameters of the GBM model		Parameters of the jump process model	
$\hat{\mu}$	0.292962	$\hat{\mu}$	0.293013
$\hat{\sigma}$	0.228361	$\hat{\sigma}$	0.178707
		$\hat{\lambda}$	19.887863
		$\hat{\mu}_Y$	0.006168
		$\hat{\sigma}_Y$	0.031290

After getting the parameters for the two models, we create 100 thousands scenarios for price process of IBM stock in one year beginning from May 1, 2012 using the GBM model (3.14) and the jump process model (3.25) respectively, and then estimate the PVaR at three confidence levels 95% , 90% , 85% with the method described in this chapter.

PVaR is an indicator for market risk over a period of time, while VaR is an indicator for risk at certain time spot. To see the difference between the two risk indicators, we calculate the VaR at the end of one year beginning from May 1, 2012.

We take the last price in each simulated scenario as the sample prices of IBM stock, and estimate VaR at three confidence levels 95% , 90% , 85% with the scenario simulation method, refer to Dowd(2002)[13] for the scenario simulation method. We omit the details of these computations here, but summarize the results in Table 3.5 below.

Table 3.5: PVaR and VaR estimated by the numerical method

Confidence levels	95 %		90 %		85 %	
	PVaR	VaR	PVaR	VaR	PVaR	VaR
GBM model	0.2469	0.1030	0.1914	0.0263	0.1593	-0.0311
Jump process model	0.2374	0.1024	0.1896	0.0334	0.1536	-0.0328

We see from the above computing experiments that risk during a period of time is much bigger than the risk at the end of that period, which signifies that PVaR is a proper alternative when risk within a period of time is of concern.

We employ MATLAB to solve the equation (3.13) for three different confidence levels 95%, 90%, 85% , and determine the corresponding PVaR for each confidence level. The value of μ, σ showed in Table 3.4. Compare the results with the simulation method are summarized in Table 3.6.

Table 3.6: PVaR of holding the IBM stock for one year

Confidence levels	95%	90%	85%
PVaR calculated by the Analytic method	0.2541	0.1999	0.1669
PVaR calculated by the Simulation method	0.2469	0.1914	0.1593

We can see that PVaR estimated with the two methods is almost the same. The slight differences are believed to be caused by computation error in simulation.

3.5 Summary

This chapter proposed a numerical method to estimate PVaR when only one risk factor is involved in investments. The numerical method was brought into shape when the risk factor is modeled as a geometric Brownian motion or a jump process. The numerical method proposed in this chapter provides a usable way for estimating PVaR, making PVaR operational in investment practice.

To illustrate the proposed numerical method, we did PVaR estimation experiments by taking an investment to IBM stock as an example, under each of the two assumptions about the stock price, our computing experiments showed that the proposed method is usable.

We also compared the risk over a period of time and the risk at certain time spot, our computing results showed that PVaR within one year is much bigger than VaR at the end of one year, signifying that PVaR is a proper alternative when risk within a period of time is of concern.

We calculated PVaR by the analytic method and the simulation method, respectively. The results show that PVaR estimated with the two methods is almost the same. Hence, the simulation method can be used to estimate PVaR.

This chapter only considered the computation of PVaR of an investment where only one risk factor is involved, we are continuing to explore methods for computing PVaR where multiple risk factors are involved, and portfolio selection with PVaR as the risk indicator.

Chapter 4

Estimation of Period Value at Risk: Multiple risk factors case

In this chapter we propose a method for estimating PVaR, for the case where there are multiple risk factors. Section 4.1 estimates PVaR by focusing on a simple case in which there are two risk factors. Section 4.2 estimates PVaR for the multiple risk factors case. Section 4.3 illustrates the proposed method in computing experiments. Section 4.4 is a summary.

Consider a portfolio consisted of multiple correlated stocks, the uncertain factors are the stock prices in the future. To estimate the PVaR of such a portfolio, we make an assumption about the values of the uncertain factors as follows:

Assumption 3 *Future price of stocks during a period can be described as a multi-dimensional geometric Brownian motion. That is, the price of stock i, denoted by $S_i(t), i = 1, 2, \cdots, n$, is the solution of the following equation,*

$$dS_i(t) = \mu_i S_i(t)dt + S_i(t) \sum_{j=1}^{m} \sigma_{ij} dB_j(t), \tag{4.1}$$

where μ_i and $\sigma_{ij}(i = 1, 2, \cdots, n; j = 1, 2, \cdots, m)$ are parameters, and $(B_1(t), B_2(t), \cdots, B_m(t))$ is a m-dimensional standard Brownian motion.

Under Assumption 3, let the purchase prices of stock i be $S_i(0), i = 1, 2, \cdots, n$, then prices of stock i can be expressed by the following formula(Nishida, 2005)[26]:

$$S_i(t) = S_i(0)exp[(\mu_i - \sum_{j=1}^{m} \sigma_{ij}^2/2)t + \sum_{j=1}^{m} \sigma_{ij} B_j(t)]. \tag{4.2}$$

Hence we can simulate the prices of stock i using (4.2), and estimate PVaR in the following steps:

Step 1. Simulate the prices of stock i, $S_i(t), t \in [0, T]; i = 1, 2, \cdots, n$

Step 2. Compute the loss of a portfolio using the prices of the stocks, let it be $L(t), t \in [0, T]$

Step 3. Compute the largest loss of $L(t), t \in [0, T]$, let it be $L_T = max\{L(t), t \in [0, T]\}$

Step 4. Repeat step 1 to step 3 N times, let L_T^k be the data of L_T in the kth time, and we have N data $\{L_T^k; k = 1, 2, \cdots, N\}$

Step 5. The $\lceil (1 - \alpha)N \rceil^{th}$ smallest value in set $\{L_T^k; k = 1, 2, \cdots, N\}$ is $PVaR_{1-\alpha}$.

So we can estimate PVaR if we can simulate the risk factors in Step 1 because computations in Step 2 ~ Step 5 are straightforward.

First, we consider a simple case where a portfolio is consisted of two correlated stocks. Then, we consider a more general case where a portfolio is consisted of multiple correlated stocks.

4.1 Estimation of PVaR: two risk factors case

A general model for m-dimensional geometric Brownian motion is given as follows:

$$\begin{cases} dS_1(t) = & \mu_1 S_1(t)dt + S_1(t)\{\sigma_{11}dB_1(t) + \sigma_{12}dB_2(t) + \cdots + \sigma_{1m}dB_m(t)\} \\ \vdots & \vdots \\ dS_i(t) = & \mu_i S_i(t)dt + S_i(t)\{\sigma_{i1}dB_1(t) + \sigma_{i2}dB_2(t) + \cdots + \sigma_{im}dB_m(t)\} \\ \vdots & \vdots \\ dS_n(t) = & \mu_n S_n(t)dt + S_n(t)\{\sigma_{n1}dB_1(t) + \sigma_{n2}dB_2(t) + \cdots + \sigma_{nm}dB_m(t)\} \end{cases} \quad (4.3)$$

where $(B_1(t), B_2(t), \cdots, B_m(t))$ is a m-dimensional standard Brownian motion, μ_i and $\sigma_{ij}(i = 1, 2, \cdots, n; j = 1, 2, \cdots, m)$ are parameters.

This section considers the case of $n = 2$. Since the number of m can be any a positive integer, we consider the following three cases:

Case A: $m = 1.(m < n)$

$$\begin{cases} dS_1(t) &= \mu_1 S_1(t)dt + S_1(t)\sigma_{11}dB(t) \\ dS_2(t) &= \mu_2 S_2(t)dt + S_2(t)\sigma_{21}dB(t) \end{cases} \quad (4.4)$$

Case B: $m = 2.(m = n)$

$$\begin{cases} dS_1(t) &= \mu_1 S_1(t)dt + S_1(t)\{\sigma_{11}dB_1(t) + \sigma_{12}dB_2(t)\} \\ dS_2(t) &= \mu_2 S_2(t)dt + S_2(t)\{\sigma_{21}dB_1(t) + \sigma_{22}dB_2(t)\} \end{cases} \quad (4.5)$$

Case C: $m = 3.(m > n)$

$$\begin{cases} dS_1(t) &= \mu_1 S_1(t)dt + S_1(t)\{\sigma_{11}dB_1(t) + \sigma_{12}dB_2(t) + \sigma_{13}dB_3(t)\} \\ dS_2(t) &= \mu_2 S_2(t)dt + S_2(t)\{\sigma_{21}dB_1(t) + \sigma_{22}dB_2(t) + \sigma_{23}dB_3(t)\} \end{cases} \quad (4.6)$$

To simulate the risk factors using these formula, it is necessary to determine their parameters first.

Case A: $m = 1.(m < n)$

Denote $S_1(t)|_{t=0}$ by $S_1(0)$, and $S_2(t)|_{t=0}$ by $S_2(0)$, then the solution of (4.4) is given by

$$\begin{cases} S_1(t) &= S_1(0)exp[(\mu_1 - \sigma_{11}^2/2)t + \sigma_{11}B(t)] \\ S_2(t) &= S_2(0)exp[(\mu_2 - \sigma_{21}^2/2)t + \sigma_{21}B(t)] \end{cases} \tag{4.7}$$

Define the profit rate of stock i, denoted by $R_i(t)$, as the logarithmic yield, according to (4.7), we have the following result.

$$\begin{cases} R_1(t) &= (\mu_1 - \sigma_{11}^2/2)t + \sigma_{11}B(t) \\ R_2(t) &= (\mu_2 - \sigma_{21}^2/2)t + \sigma_{21}B(t) \end{cases} \tag{4.8}$$

(4.8) means that the correlation coefficient of $R_1(t)$ and $R_2(t)$ is 1 at any time, which is a rare case. So m should be set to a number bigger than 1 for the case of $n = 2$.

Case B: $m = 2.(m = n)$

We use two independent Brownian motions $B_1(t)$ and $B_2(t)$ to express correlated stock prices $S_1(t)$ and $S_2(t)$

$$\begin{cases} dS_1(t) &= \mu_1 S_1(t)dt + S_1(t)\{\sigma_{11}dB_1(t) + \sigma_{12}dB_2(t)\} \\ dS_2(t) &= \mu_2 S_2(t)dt + S_2(t)\{\sigma_{21}dB_1(t) + \sigma_{22}dB_2(t)\} \end{cases} \tag{4.9}$$

Let $S_1(t)|_{t=0}$ be $S_1(0)$, $S_2(t)|_{t=0}$ be $S_2(0)$, then solution of equation (4.9) is

$$\begin{cases} S_1(t) &= S_1(0)exp[(\mu_1 - \frac{\sigma_{11}^2+\sigma_{12}^2}{2})t + \sigma_{11}B_1(t) + \sigma_{12}B_2(t)] \\ S_2(t) &= S_2(0)exp[(\mu_2 - \frac{\sigma_{21}^2+\sigma_{22}^2}{2})t + \sigma_{21}B_1(t) + \sigma_{22}B_2(t)] \end{cases} \tag{4.10}$$

In order to simulate $S_1(t)$ and $S_2(t)$ using(4.10), it is necessary to estimate the parameters in (4.10). We next suggest a method to determine these parameters using historical price data of the two stocks.

Suppose we have $D + 1$ sequential close daily prices of each stock in past T^1 years, denoted by $S_i(t_0^1)$, $S_i(t_1^1)$, $S_i(t_2^1)$, $\cdots$, $S_i(t_D^1), i = 1,2$. The elapsed time between any two sequential time spots is $\Delta t = T^1/D$. The rate of profit in two adjacent times, measured by the logarithm yield, is $r_i(t_d^1) = ln\frac{S_i(t_d^1)}{S_i(t_{d-1}^1)}, d = 1, 2, \cdots, D, i = 1, 2$.

Define the following notations:

E_i: expectation of $r_i(t_d^1), d = 1, 2, \cdots, D, i = 1, 2$.

V_i: variance of $r_i(t_d^1), d = 1, 2, \cdots, D, i = 1, 2$.

C_{12}: covariance of $r_1(t_d^1)$ and $r_2(t_d^1), d = 1, 2, \cdots, D$

Denote two adjacent points of time in $[0, T]$ with time interval Δt by t_{d-1} and t_d, and the profit rate in the two adjacent times measured by the logarithm yield by $r_i(t_d), i = 1, 2$. Because the stock prices are supposed to be a 2-dimensional geometric Brownian motion as described by (4.10), we have the following relationship:

$$\begin{cases} r_1(t_d) = & (\mu_1 - \frac{\sigma_{11}^2+\sigma_{12}^2}{2})(t_d - t_{d-1}) \\ & +\sigma_{11}(B_1(t_d) - B_1(t_{d-1})) + \sigma_{12}(B_2(t_d) - B_2(t_{d-1})) \\ r_2(t_d) = & (\mu_2 - \frac{\sigma_{21}^2+\sigma_{22}^2}{2})(t_d - t_{d-1}) \\ & +\sigma_{21}(B_1(t_d) - B_1(t_{d-1})) + \sigma_{22}(B_2(t_d) - B_2(t_{d-1})) \end{cases} \tag{4.11}$$

Because $(B_1(t), B_2(t))$ is a 2-dimensional Brownian motion, according to the properties of standard Brownian motion, we know that $B_j(t_d) - B_j(t_{d-1}) \sim N(0, \Delta t), j = 1, 2$. Hence, the expectation, variance and covariance of the profit rates of the two stocks is $(\mu_1 - \frac{\sigma_{11}^2+\sigma_{12}^2}{2})\Delta t$, $(\mu_2 - \frac{\sigma_{21}^2+\sigma_{22}^2}{2})\Delta t$, $(\sigma_{11}^2 + \sigma_{12}^2)\Delta t$, $(\sigma_{21}^2 + \sigma_{22}^2)\Delta t$, and $(\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22})\Delta t$, respectively.

On the other hand, using historical price of the two stocks, we can calculate the expectation, variance and covariance of the profit rates. One way to determine

μ_1, μ_2 and $\sigma_{ij}(i = 1,2; j = 1,2)$ is to make the values of expectation, variance and covariance calculated from (4.11) consistent with that estimated using the historical price data.

That is, we determine these parameters from solving the following equations:

$$\begin{cases} (\mu_1 - \frac{\sigma_{11}^2+\sigma_{12}^2}{2})\Delta t &= E_1 \\ (\mu_2 - \frac{\sigma_{21}^2+\sigma_{22}^2}{2})\Delta t &= E_2 \end{cases} \tag{4.12}$$

$$\begin{cases} (\sigma_{11}^2 + \sigma_{12}^2)\Delta t &= V_1 \\ (\sigma_{21}^2 + \sigma_{22}^2)\Delta t &= V_2 \\ (\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22})\Delta t &= C_{12} \end{cases} \tag{4.13}$$

It is easy to get μ_1 and μ_2 from (4.12) and (4.13)

$$\begin{cases} \mu_1 &= \frac{E_1+0.5V_1}{\Delta t} \\ \mu_2 &= \frac{E_2+0.5V_2}{\Delta t} \end{cases} \tag{4.14}$$

Equations (4.13) have 4 variables and 3 equations, its solution is not unique. One way to solve this equations is using the Cholesky decomposition. Let

$$A = \begin{bmatrix} \frac{V_1}{\Delta t} & \frac{C_{12}}{\Delta t} \\ \frac{C_{12}}{\Delta t} & \frac{V_2}{\Delta t} \end{bmatrix} = \begin{bmatrix} \sigma_{11}^2 + \sigma_{12}^2 & \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22} \\ \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22} & \sigma_{21}^2 + \sigma_{22}^2 \end{bmatrix}$$

$$B = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

Since, equations(4.13) can be written as

$$BB' = A. \tag{4.15}$$

If matrix A has real entries and is symmetric(or more generally, has complex-valued entries and is Hermitian) and positive definite, then matrix A can be decomposed as the product of two matrixes BB', where B is a lower triangular matrix with strictly positive diagonal entries, and B' denotes the conjugate transpose of B. This is the Cholesky decomposition, refer to Wilkinson(1965)[35] for details.

Because matrix A defined in this section is symmetric, and $|A| = \frac{V_1V_2}{\Delta t^2} - \frac{C_{12}^2}{\Delta t^2} = \frac{V_1V_2(1-\rho^2)}{\Delta t^2} \geq 0$, where ρ is the correlation coefficient of the profit rates, $\rho = \frac{C_{12}}{\sqrt{V_1V_2}}$, we know that matrix A is Cholesky decomposable. After getting the Cholesky decomposition of matrix A, we get matrix B, thus getting a solution of (4.13).

Case C: $m = 3.(m > n)$

We know from case B that using two independent Brownian motion can express two correlated $S_1(t)$ and $S_2(t)$, and the parameters in (4.9) can be estimated without producing errors in most cases. It is not necessary to use more than two independent Brownian motions to express two correlated $S_1(t)$ and $S_2(t)$.

Simulation of stock prices

To simulate the stock price described with formula (4.10) wherein the 2-dimensional standard BM is included, we first simulate $R_i(t) = ln\frac{S_i(t)}{S_i(0)}$. If we can simulate $R_i(t)$, then stock prices can be simulated using the following relation:

$$S_i(t) = S_i(0)e^{R_i(t)}. \tag{4.16}$$

According to formula (4.10), we have,

$$R_i(t) = (\mu_i - \frac{\sigma_{i1}^2 + \sigma_{i2}^2}{2})t + \sigma_{i1}B_1(t) + \sigma_{i2}B_2(t) \tag{4.17}$$

Since $(B_1(t), B_2(t))$ is a 2-dimensional standard BM, the following expression is an equivalent formula for $R_i(t)$,

$$R_i(t) = (\mu_i - \frac{\sigma_{i1}^2 + \sigma_{i2}^2}{2})t + \sigma_{i1}\sqrt{t}\xi_1 + \sigma_{i2}\sqrt{t}\xi_2, \tag{4.18}$$

where (ξ_1, ξ_2) is a 2-dimensional variable following the 2-dimensional standard normal distribution.

Because many software are available for simulating the standard normal distribution, we can easily simulate $R_i(t)$ using (4.18), then simulate $S_i(t)$ using (4.16).

Calculation of PVaR

After producing the stock prices in each scenario, we then calculate the loss rate in each simulated scenario using the following formula,

$$L(\omega^k(t)) = -\frac{\sum_{i=1}^2 b_i S_i^k(t) - \sum_{i=1}^2 b_i S_i(0)}{\sum_{i=1}^2 b_i S_i(0)}, t \in [0, T]; k = 1, 2, \cdots, N. \quad (4.19)$$

where $S_i^k(t)$ is the price of stock i at time t on the k^{th} simulated scenario, b_i is the share of stock i in the portfolio.

Then the largest loss rate in the k^{th} scenario is given by

$$L_T^k = max\{L(\omega^k(t)); t \in [0, T]\}.$$

According to the definition of PVaR, the $\lceil(1-\alpha)N\rceil^{th}$ smallest value in $\{L_T^k, k = 1, 2, \cdots, N\}$ is $PVaR_{1-\alpha}$. Thus we can get the PVaR of a portfolio by simulating its components' price processes, and calculate it as stated above.

4.2 Estimation of PVaR: Multiple risk factors case

Under Assumption 3, with the initial condition $S_i(t)|_{t=0} = S_i(0), i = 1, 2, \cdots, n$, the solution of (4.1) is given by,

$$\begin{cases} S_1(t) = & S_1(0)exp[(\mu_1 - \frac{\sigma_{11}^2+\cdots+\sigma_{1j}^2+\cdots+\sigma_{1m}^2}{2})t \\ & +\sigma_{11}B_1(t) + \cdots + \sigma_{1j}B_j(t) + \cdots + \sigma_{1m}B_m(t)] \\ \cdots & \cdots \\ S_i(t) = & S_i(0)exp[(\mu_i - \frac{\sigma_{i1}^2+\cdots+\sigma_{ij}^2+\cdots+\sigma_{im}^2}{2})t \\ & +\sigma_{i1}B_1(t) + \cdots + \sigma_{ij}B_j(t) + \cdots + \sigma_{im}B_m(t)] \\ \cdots & \cdots \\ S_n(t) = & S_n(0)exp[(\mu_n - \frac{\sigma_{n1}^2+\cdots+\sigma_{nj}^2+\cdots+\sigma_{nm}^2}{2})t \\ & +\sigma_{n1}B_1(t) + \cdots + \sigma_{nj}B_j(t) + \cdots + \sigma_{nm}B_m(t)] \end{cases} \tag{4.20}$$

In order to simulate $S_i(t), i = 1, 2, \cdots, n$ using(4.20), it is necessary to estimate the parameters in (4.20). We next suggest a method to determine these parameters using historical price data of these stocks.

Suppose we have $D + 1$ sequential close daily prices of each stock in past T^1 years, denoted by $S_i(t_0^1)$, $S_i(t_1^1)$, $S_i(t_2^1)$, $\cdots$, $S_i(t_D^1), i = 1, 2, \cdots, n$. The elapsed time between any two sequential time spots is $\Delta t = T^1/D$. The rate of profit in two adjacent times, measured by the logarithm yield, is $r_i(t_d^1) = ln\frac{S_i(t_d^1)}{S_i(t_{d-1}^1)}, d = 1, 2, \cdots, D; i = 1, 2, \cdots, n$.

Hence we can estimate parameters of formula (4.20) in the following steps:

Step 1. Compute daily profit rates of each stock using historical prices of the stock, denoted by $r_i(t_d^1), i = 1, 2, \cdots, n, d = 1, 2, \cdots, D$

Step 2. Compute the expectation, variance and covariance of these daily profit rates, $r_i(t_d^1)$, let them be E_i, V_i and $C_{i_1 i_2}, i = 1, 2, \cdots, n; i_1, i_2 = 1, 2, \cdots, n, i_1 \neq i_2$

Step 3. Compute the theoretical values of expectation, variance and covariance of these daily profit rates by formula (4.20), denoted by $E(r_i(t_d))$, $V(r_i(t_d))$ and $C(r_{i_1}(t_d), r_{i_2}(t_d))$, where $r_i(t_d) = ln\frac{S_i(t_d)}{S_i(t_{d-1})}, d = 1, 2, \cdots, D; i, i_1, i_2 = 1, 2, \cdots, n, i_1 \neq i_2$.

The theoretical values of expectation, variance and covariance of the profit rates of the $r_i(t_d)$ are

$$\begin{cases} E(r_i(t_d)) & = (\mu_i - \frac{\sum_{j=1}^m \sigma_{ij}^2}{2})\Delta t \\ V(r_i(t_d)) & = \Delta t \sum_{j=1}^m \sigma_{ij}^2 \\ C(r_{i_1}(t_d), r_{i_2}(t_d)) & = \Delta t \sum_{j=1}^m \sigma_{i_1 j}\sigma_{i_2 j} \end{cases} \tag{4.21}$$

It is desirable that the theoretical values of the expectation, the variance and covariance of the daily profit are consistent with that estimated using historical price data, that is, the parameters in (4.20) have better to satisfy the following equations

$$(\mu_i - \frac{\sum_{j=1}^m \sigma_{ij}^2}{2})\Delta t = E_i, \ i = 1, 2, \cdots, n \tag{4.22}$$

$$\begin{array}{lcl} \Delta t \sum_{j=1}^m \sigma_{ij}^2 & = & V_i, \ i = 1, 2, \cdots, n \\ \Delta t \sum_{j=1}^m \sigma_{i_1 j}\sigma_{i_2 j} & = & C_{i_1 i_2}, \ i_1, i_2 = 1, 2, \cdots, n, i_1 \neq i_2 \end{array} \tag{4.23}$$

Let

$$A = \begin{bmatrix} \frac{V_1}{\Delta t} & \frac{C_{12}}{\Delta t} & \cdots & \frac{C_{1n}}{\Delta t} \\ \frac{C_{12}}{\Delta t} & \frac{V_2}{\Delta t} & \cdots & \frac{C_{2n}}{\Delta t} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{C_{1n}}{\Delta t} & \frac{C_{2n}}{\Delta t} & \cdots & \frac{V_n}{\Delta t} \end{bmatrix},$$

$$B = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1m} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nm} \end{bmatrix}$$

If the matrix A is symmetric and positive definite, we can get a solution of B by doing Cholesky decomposition, as in section 4.1.

If A can not be decomposed, we determine the parameters $\sigma_{ij}(i = 1, 2, \cdots, n, j = 1, 2, \cdots, m)$ by minimizing the difference between the theoretical value and historical value of the variance and covariance, that is, we determine the parameters by solving the following minimization model,

$$min \sum_{i=1}^{n}(V_i - V(r_i(t_d)))^2 + \sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1-1}(C_{i_1 i_2} - C(r_{i_1}(t_d), r_{i_2}(t_d)))^2 \tag{4.24}$$

where $V(r_i(t_d))$ and $C(r_{i_1}(t_d), r_{i_2}(t_d))$ is the theoretical value of the variance and covariance, and V_i and $C_{i_1 i_2}$ is the historical value.

From equations (4.22) and (4.23) we have

$$\mu_i = \frac{0.5V_i + E_i}{\Delta t}, i = 1, 2, \cdots, n.$$

Simulation of stock prices

To simulate the stock price described with formula (4.20) wherein the m-dimensional standard BM is included, we first simulate $R_i(t) = ln\frac{S_i(t)}{S_i(0)}$. If we can simulate $R_i(t)$, then stock prices can be simulated using the following relation:

$$S_i(t) = S_i(0)e^{R_i(t)}. \tag{4.25}$$

According to formula (4.20), we have,

$$R_i(t) = (\mu_i - \frac{\sum_{j=1}^{m}\sigma_{ij}^2}{2})t + \sum_{j=1}^{m}\sigma_{ij}B_j(t). \tag{4.26}$$

Since $(B_1(t), \cdots, B_m(t))$ is a m-dimensional standard BM, the following expression is an equivalent formula for $R_i(t)$,

$$R_i(t) = (\mu_i - \frac{\sum_{j=1}^{m}\sigma_{ij}^2}{2})t + \sum_{j=1}^{m}\sigma_{ij}\sqrt{t}\xi_j, \tag{4.27}$$

where $(\xi_1, \cdots, \xi_m)$ is a m-dimensional variable following the m-dimensional standard normal distribution.

Because many software are available for simulating the standard normal distribution, we can easily simulate $R_i(t)$ using (4.27), then simulate $S_i(t)$ using (4.25).

Calculation of PVaR

After producing the stock prices in each scenario, we then calculate the loss rate in each simulated scenario using the following formula,

$$L(\omega^k(t)) = -\frac{\sum_{i=1}^n b_i S_i^k(t) - \sum_{i=1}^n b_i S_i(0)}{\sum_{i=1}^n b_i S_i(0)}, t \in [0, T]; k = 1, 2, \cdots, N. \quad (4.28)$$

where $S_i^k(t)$ is the price of stock i at time t on the k^{th} simulated scenario, b_i is the share of stock i in the portfolio.

Then the largest loss rate in the k^{th} scenario is given by

$$L_T^k = max\{L(\omega^k(t)); t \in [0, T]\}.$$

According to the definition of PVaR, the $\lceil(1 - \alpha)N\rceil^{th}$ smallest value in $\{L_T^k, k = 1, 2, \cdots, N\}$ is $PVaR_{1-\alpha}$. Thus we can get the PVaR of a portfolio by simulating its components' price processes, and calculate it as stated above.

4.3 Computation experiments

In order to illustrate the methods proposed in previous section for estimating PVaR, this section does computation experiments using the historical price data of Dow's 30 stocks in 2011, which was downloaded from Yahoo!Finance. The components for Dow Jones industrial average is shown in Figure 4.1.

Traded	Name	Founded	Headquarters	Industry	Products
AXP	American Express	1850	New York	Banking, Financial	Charge card, credit cards, traveler's
BA	The Boeing Company	1916	Illinois	Aerospace, Defense	737/777/787/F/A-18E/F Super Hornet/CH-47 Chinook/702
CAT	Caterpillar Inc.	1925	Illinois	Heavy equipment, Engines, Financial	D11 Bulldozer, 345C L Excavator, 930G wheel loader
CSCO	Cisco Systems, Inc.	1984	California	Networking equipment	Networking Device, Network Management,Cisco IOS and NX-OS
CVX	Chevron Corporation	1984	California	Oil and gas	Petroleum, natural gas and other petrochemicals, See Chevron products
DD	E. I. du Pont de Nemours and	1802	Delaware	Chemicals	Corian, Delrin, Kevlar, Mylar
DIS	The Walt Disney Company	1923	California	Mass media	Cable television, publishing, movies, theme parks, broadcasting, radio, web
GE	General Electric Company	1892	Connecticut	Conglomerate	Appliances, aviation, consumer electronics, electrical distribution, electric motors, energy, finance, gas, healthcare, lighting, locomotives, oil,
GS	The Goldman Sachs Group, Inc.	1869	New York	Banking, financial services	Asset management, commercial banking, commodities, investment banking, investment management,
HD	The Home Depot, Inc.	1978	Georgia	Retailing	Home appliances, tools, hardware, lumber, building materials, paint, plumbing, flooring, garden supplies &
IBM	International Business Machines Corporation	1911	New York	Computer hardware, Computer software, IT services, IT consulting	See IBM products
INTC	Intel Corporation	1968	California	Semiconductors	Bluetooth chipsets, flash memory, microprocessors, motherboard
JNJ	Johnson & Johnson	1886	New jersey	Medical equipment pharmaceutical	See list of Johnson & Johnson products
JPM	JPMorgan Chase & Co.	2000	New York	Banking, Financial services	Consumer banking, corporate banking, credit cards, finance and insurance, foreign currency exchange,
KO	The Coca-Cola	1886	Georgia	Beverage	List of The Coca-Cola Company
MCD	McDonald's	1940	Illinois	Restaurants	Fast food
MMM	Minnesota Mining and Manufacturing	1902	Minnesota	Conglomerate	List of 3M Company products
MRK	Merck & Co. Inc.	1891	New jersey	Pharmaceuticals	Gardasil/Singulair/Propecia/Proscar/Zocor Vioxx Fosamax
MSFT	Microsoft Corporation	1975	Washington	Computer software	Windows (Phone, Server)/ Office/Dynamics/Azure/Xbox/Surface
NKE	Nike, Inc.	1964	Oregon	Apparel accessories	Athletic footwear and apparel, sport equipments and other athletic and
PFE	Pfizer Inc.	1849	New York	Pharmaceutical	Pharmaceutical products
PG	The Procter & Gamble Company	1837	Ohio	Consumer goods	Foods, beverages, cleaning agents and personal care products
T	AT&T, Inc.	1983	Texas	Telecommunications	Fixed line and mobile telephony, broadband and fixed-line internet
TRV	The Travelers	1853	New York	Insurance	Insurance policies, Risk management
UNH	UnitedHealth Group Incorporated	1977	Minnesota	Managed health care	Uniprise, Golden Rule,[1] Health Care Services, Specialized Care Services,
UTX	United Technologies Corporation	1975	Connecticut	Conglomerate	researches, develops, and manufactures high-technology
V	Visa Inc.	1958	California	Financial services	Credit cards, payment systems
VZ	Verizon Communications Inc.	1983	New York	Telecommunications	Fixed-line and mobile telephony, broadband and fixed-line internet services, digital television and
WMT	Wal-Mart Stores Inc.	1962	Arkansas	Retail	Apparel/footwear specialty, cash & carry/warehouse club, discount store, hypermarket/supercenter/superstore,
XOM	Exxon Mobil	1999	Texas	Oil and gas	Fuels, lubricants, petrochemicals

Figure 4.1: The Components of Dow Jones Industrial Average

The historical stock prices of these stocks and the correlation coefficients in 2011 are shown in appendix C.

With the historical price data, the daily profit rate of 30 stocks can be calculated easily. Here, we use a portfolio consisted of 5 stocks: AXP, BA, CAT, CSCO and CVX, the investment period is one year.

Since we consider a portfolio with 5 stocks, we know that 5 independent Brownian motions are enough to express the future prices of the five stocks. However, if using 4 or 3 independent Brownian motions we have less parameters in(4.20), solving (4.24) may cause a bigger error.

To compare the difference of using different m, we do computing experiments solving (4.24) by $m = 4$ and $m = 5$, and compare the errors. The results are shown in Table4.1.

Define the absolute value of the deviation between the theoretical value and the historical value of variances or covariances as Error1:

$$Error1 = max|(C_{i_1i_2} - C(r_{i_1}(t_d), r_{i_2}(t_d)))|, i_1, i_2 = 1, 2, \cdots, n. \tag{4.29}$$

where $C(r_{i_1}(t_d), r_{i_2}(t_d))$ and $C_{i_1i_2}$is the theoretical value and the historical value of covariance of daily profit rate of stock i_1 and stock i_2. $C_{i_1i_2} = V_i$ and $C(r_{i_1}(t_d), r_{i_2}(t_d)) = V(r_i(t_d))$ when i_1 equals i_2, $C(r_{i_1}(t_d), r_{i_2}(t_d)) = \Delta t\Sigma_{j=1}^{m}\hat{\sigma}_{i_1j}\hat{\sigma}_{i_2j}$.

Define the sum of squared deviation between the theoretical value and the historical value of the variance and the covariance of the daily profit rate of stock i_1 and stock i_2 as Error2:

$$Error2 = \sum_{i_1=1}^{n}\sum_{i_2=1}^{i_1}(C_{i_1i_2} - C(r_{i_1}(t_d), r_{i_2}(t_d)))^2 \tag{4.30}$$

Table 4.1: Comparison of the theoretical value and the historical value of variances and covariances

	Historical value	Theoretical value($\hat{\sigma}_{ij}$)	
	$n = 5$	m	
		4	5
V_1	0.37079988	0.37079988	0.37079988
V_2	0.37068196	0.37068196	0.37068196
V_3	0.55838386	0.55838386	0.55838386
V_4	0.49133711	0.49133711	0.49133711
V_5	0.30623518	0.21056768	0.30623518
C_{12}	0.27685939	0.27685939	0.27685939
C_{13}	0.33529342	0.33529342	0.33529342
C_{14}	0.23600072	0.23600072	0.23600072
C_{15}	0.23706202	0.23706202	0.237062016
C_{23}	0.35066324	0.35066324	0.350663243
C_{24}	0.23137157	0.23137157	0.231371572
C_{25}	0.24316816	0.24316816	0.243168165
C_{34}	0.30008957	0.30008958	0.30008958
C_{35}	0.33017391	0.33017391	0.33017391
C_{45}	0.22426830	0.22426830	0.22426830
Error1		0.02401254047	6.9388939039e-018
Error2		5.76602099823e-004	4.814824861e-035

It was clear that the error of parameters calculated by $m = 5$ is smaller than calculated by $m = 4$. And in general, the matrix A in section 4.2 can be Cholesky decompose. Since, in this study we use Cholesky decomposition to decompose the correlation matrix for determine the parameters, and let the m is equal to n, without special instructions.

Estimation of PVaR

We estimate the risk of a portfolio consisted of five stocks: CAT, CVX, DD, 3M, UTX. We use the daily close prices of these stocks in 2011 (these data in the Appendix C) to estimate the parameters in model (4.20), and get the following result:

Table 4.2: Parameters estimation

$\hat{\mu}_i$	$\hat{\sigma}_{ij}$				
0.055456556499989	0.374282298539456	0	0	0	0
0.223188552006711	0.221447411304994	0.166951567946334	0	0	0
-0.005824500228372	0.261610134580743	0.063309331564716	0.148360343639164	0	0
0.010613984919450	0.226122607119851	0.059658363084100	0.040597837927789	0.149508754452492	0
-0.011989222740074	0.236019136492254	0.049125554480826	0.035116426579900	0.058924019379410	0.123805113588610
*Error*1		2.775557561562891e-017	*Error*2		9.629649721936179e-034

We calculate the PVaR at different confidence level, 95% , 90% and 85% for this portfolio for one year beginning from 2013/8/1. We create 100 thousands samples for each price process of the five component stocks, and estimate the PVaR with the method descried in previous, and the results are shown in Table 4.3.

To compare the risks measured by different measures, we calculate the VaR at the same confidence level of this portfolio at the end of one year beginning from

Table 4.3: PVaR and VaR estimated by the numerical method

Confidencelevels	95%	90%	85%
PVaR	0.217659703623563	0.179354294306515	0.153668861213479
VaR	0.170007432907625	0.120184761501073	0.088032486699086

2013/8/1. We use the last price of each simulated price process as the price sample of the corresponding stock at end of this period, thus we get 100 thousands price samples for each component stock of the portfolio. Then we calculate the loss of the portfolio using these price data, producing 100 thousands samples of the portfolio loss, then VaR is estimated using the scenario simulation method, the corresponding VaR of the PVaR, in Table4.3.

We can see from the above computing experiment that risk during a period of time is much bigger that the risk at the end of that period. VaR does not reflect the risk within a time span, PVaR is a proper alternative when risk within a period of time that is of concern.

4.4 Summary

This chapter proposed a numerical method to estimate PVaR when multiple risk factors are involved in investments.

The numerical method was brought into shape when the risk factors are modeled as a multi-dimensional geometric Brownian motion. The numerical method proposed in this chapter provides a usable way for estimating PVaR, making PVaR operational in investment practice. To illustrate the proposed numerical method, we did computing experiments.

This chapter considered the computation of PVaR of a portfolio consisted of multiple risk factors, which is needed in exporting portfolio selection problems, when risk is measured by PVaR.

Chapter 5

Portfolio Selection based on Period Value at Risk

This chapter aims at formulating portfolio selection problems with PVaR as the indicator of market risk, and proposing resolution methods for the models built. The portfolio selection problems are explained in Section 5.1. In Section 5.2 we formulate the portfolio selection problems by minimizing the risk of investment, and propose a method for solving the model built. Section 5.3 formulates portfolio selection problems by maximizing the return of investment, and proposes a method for solving the model built. Section 5.4 is a summary of this chapter.

5.1 Portfolio selection and its formulation

We consider the portfolio selection problem in this chapter, it is based on a single period model of investment. At the beginning of a period, an investor allocates the capital among various securities, and may hold the securities for a period of time. We suppose the maximum of the period is known.

Here, we consider the portfolio selection problem with n financial instruments.

Let $x = (x_1, x_2, \cdots, x_n)$ denote an n-dimensional vector whose components represent the percentage of wealth allocated to the financial investments. How to determine x so that the performance of x will best meet investors' requirement is called the portfolio selection problem.

Following the seminal work by Markowitz(1952)[23], the portfolio selection problem is modeled as a return-risk bicriteria optimization problem.

That means, investors evaluate investment alternatives by their return and risk, denoted by $Return(x)$ and $Risk(x)$, respectively. The portfolio selection problems are usually formulated by one of the following two optimization models.

$$\begin{array}{ll} min & Risk(x) \\ s.t. & Return(x) \geq r_0 \\ & \sum_{i=1}^{n} x_i = 1; 0 \leq x_i \leq 1 \end{array}$$

or

$$\begin{array}{ll} max & Return(x) \\ s.t. & Risk(x) \leq risk_0 \\ & \sum_{i=1}^{n} x_i = 1; 0 \leq x_i \leq 1 \end{array}$$

where r_0 indicates the desired return target for investors aiming at minimizing risk, and $risk_0$ indicates the acceptable risk for investors aiming at maximizing return.

In the original Markowitz model(Markowitz, 1952)[23] the risk was measured by the standard deviation or variance of profit rate. In this study, risk is measured by Period Value-at-Risk.

5.2 Risk minimization: model and its solution method

We use PVaR as the risk indicator, then the risk minimization model for portfolio selection problems is given as follows:

$$\begin{array}{ll} min & PVaR_{1-\alpha}(x) \\ s.t. & Return(x) \geq r_0 \\ & \sum_{i=1}^{n} x_i = 1; 0 \leq x_i \leq 1 \end{array} \tag{5.1}$$

where $x = (x_1, x_2, \cdots, x_n)$ is the decision variable, x_i denotes the percentage of wealth allocated to asset i; r_0 indicates the desired return target for investors.

Under Assumption 3 in Chapter 4, future price of stocks during the period can be described as a multi-dimensional geometric Brownian motion, we can use the simulation method to estimate PVaR.

To estimate PVaR for a portfolio consisted of n stocks, we simulate these stock prices over time span $[0, T]$ for N times, producing 252 close daily prices for each stock in one year. Let D be the number of time spots in the maximum investment period $[0, T]$ where stock prices are simulated, and time interval of two adjacent time spots be Δt.

We use the following notations:

S_{it}^k: price of stock i at time t in the k^{th} scenario, $t \in [0, T]$

S_{i0}: purchase price of stock i

r_{it}^k: profit rate of stock i at time t in the k^{th} scenario, $r_{it}^k = \frac{S_{it}^k - S_{i0}}{S_{i0}}$

r_i: expectation of r_{it}^k, $r_i = \frac{1}{ND} \sum_{k=1}^{N} \sum_{t=1}^{D} r_{it}^k$; $i = 1, 2, \cdots, n$

$R_t^k(x)$: profit rate of portfolio at time t in the k^{th} scenario

$L_t^k(x)$: loss rate of portfolio at time t in the k^{th} scenario

$L_T^k(x)$: largest loss rate of portfolio over time span $[0, T]$, in the k^{th} scenario

Then we have the following formulas.

$$R_t^k(x) = \sum_{i=1}^{n} r_{it}^k x_i;\ t = 1, 2, \cdots, D;\ k = 1, 2, \cdots, N.$$

$$L_t^k(x) = -R_t^k(x); \qquad L_T^k(x) = max\{L_t^k(x); t = 1, 2, \cdots, D\},$$

the $\lceil (1-\alpha)N \rceil^{th}$ smallest value in set $\{L_T^k(x); k = 1, 2, \cdots, N\}$ is $PVaR_{1-\alpha}$. According to the definition of PVaR, for example

$$PVaR_{0.95}(x) = \{L_T^k(x); k = 1, 2, \cdots, N\}^{\lceil 0.95N \rceil}.$$

Hence, model (5.1) can be rewritten as follows:

$$\begin{array}{lll} \min & \{L_T^k(x); k = 1, 2, \cdots, N\}^{\lceil (1-\alpha)N \rceil} & (a) \\ \text{s.t.} & \sum_{i=1}^{n} r_i x_i \geq r_0 & \\ & L_T^k(x) = max\{L_t^k(x); t = 1, 2, \cdots, D\};\ k = 1, 2, \cdots, N & (b) \\ & \sum_{i=1}^{n} x_i = 1;\ 0 \leq x_i \leq 1 & \end{array} \qquad (5.2)$$

Because of the collation function in the objective function and the maximum function in constraint condition, solving model (5.2) directly is difficult. We will suggest to solve this model by changing it into an equivalent model in the next subsection.

5.2.1 A solution method

We first prove that the following model is an equivalent model of model (5.2).

$$
\begin{array}{lll}
\min & Z & (a1) \\
\text{s.t.} & \sum_{i=1}^{n} r_i x_i \geq r_0 & \\
& Z^k \geq L_t^k(x);\ t = 1,2,\cdots,D; k = 1,2,\cdots,N & (b1) \\
& y_k = \begin{cases} 0, & Z \geq Z^k \\ 1, & Z < Z^k \end{cases} \quad k = 1,2,\cdots,N & (b2) \\
& & (b3) \\
& \sum_{k=1}^{N} y_k \leq \lfloor \alpha N \rfloor & (b4) \\
& \sum_{i=1}^{n} x_i = 1;\ 0 \leq x_i \leq 1 &
\end{array}
\quad (5.3)
$$

where Z^k is an intermediate variable, expressing the largest loss of portfolio over time span $[0, T]$ in the k^{th} scenario, and y_k is an intermediate variable, recording the number of the largest losses bigger than Z.

Proposition 2 *Models (5.2) and (5.3) are equivalent if they have the same optimal solution.*

Proof

We define the following notations.

x^*: optimal solution of model(5.2).

$PVaR^*_{1-\alpha}$: optimal value of model(5.2).

x': optimal solution of model(5.3).

$PVaR'_{1-\alpha}$: optimal value of model(5.3).

(1) We prove that x^* is the optimal solution of model (5.3). Let $Z^k = L_T^k(x^*)$, and $Z = PVaR^*_{1-\alpha}$. Because x^* is the optimal solution of model (5.2), $L_T^k(x^*) = max\{L_t^k(x^*); t = 1,2,\cdots,D\}$, $Z^k \geq L_t^k(x^*)$ for all t and k in model (5.3), and $PVaR^*_{1-\alpha}$ as the optimal value of model (5.2), hence, the number of Z^k bigger than $PVaR^*_{1-\alpha}$ is less than or equal $\lfloor \alpha N \rfloor$. Since, all constraints are satisfied at

$(x^*, PVaR^*_{1-\alpha})$ in model (5.3), x^* is a feasible solution of model (5.3), with the objective value $PVaR^*_{1-\alpha}$.

Suppose that x^* is not an optimal solution of model (5.3). Because $PVaR'_{1-\alpha}$ is an optimal value of model (5.3), $PVaR'_{1-\alpha} \leq PVaR^*_{1-\alpha}$ holds.

The optimal solution of model (5.3) x' satisfies all constraints in model (5.2), and the objective value is the $\lceil(1-\alpha)N\rceil^{th}$ smallest value in the $\{L^k_T(x'); k = 1, 2, \cdots, N\}$, denote by $PVaR'^{(5.2)}_{1-\alpha}$.

From constraint (b1) for x', there are two possible $Z^k > L^k_T(x')$ or $Z^k = L^k_T(x')$. $PVaR'^{(5.2)}_{1-\alpha}$ is the $\lceil(1-\alpha)N\rceil^{th}$ smallest value in the $\{L^k_T(x'); k = 1, 2, \cdots, N\}$, and $PVaR'_{1-\alpha}$ is the $\lceil(1-\alpha)N\rceil^{th}$ smallest value in the $\{Z^k; k = 1, 2, \cdots, N\}$, hence, $PVaR'^{(5.2)}_{1-\alpha} \leq PVaR'_{1-\alpha}$ holds.

Because $PVaR'_{1-\alpha}$is the optimal value of model (5.3), $PVaR'^{(5.2)}_{1-\alpha} = PVaR'_{1-\alpha}$ holds.

That means, there is a feasible solution of model (5.2), x' , with the objective value $PVaR'_{1-\alpha}$ is less than the optimal value $PVaR^*_{1-\alpha}$. Therefore, the assumption that $PVaR^*_{1-\alpha}$ is the optimal value of model (5.2) is contradicted. Hence x^* is an optimal solution of model (5.3), with the optimal value $PVaR^*_{1-\alpha}$.

(2) We prove that x' is an optimal solution of model (5.2). Because x' is an optimal solution of model (5.3), it satisfies all constraints of model (5.2), and the objective value is the $\lceil(1-\alpha)N\rceil^{th}$ smallest value in the $\{L^k_T(x'); k = 1, 2, \cdots, N\}$, denote by $PVaR'^{(5.2)}_{1-\alpha}$. By constraint (b1) has $Z^k \geq L^k_T(x')$, $PVaR'_{1-\alpha} \geq PVaR'^{(5.2)}_{1-\alpha}$ holds. And $PVaR'_{1-\alpha}$ is the optimal value of model (5.3), hence $PVaR'_{1-\alpha} = PVaR'^{(5.2)}_{1-\alpha}$ holds.

If x' is not an optimal solution of model (5.2), $PVaR'_{1-\alpha} > PVaR^*_{1-\alpha}$ holds.

We know the optimal solution of model (5.2) x^* is a feasible solution of model (5.3). That means, the assumption that $PVaR'_{1-\alpha}$ is the optimal value of model (5.3) is contradicted. Hence, the optimal solution of model (5.3) is also the optimal solution in model (5.2).

This ends the proof.

The following proposition states that the optimal solution of model(5.3) can be obtained by solving a mixed integer programming.

Proposition 3 *Let (x^*, y^*, Z^*) be the unique optimal solution of the following model, then Models (5.3) reaches the optimum at x^* and $Z^* = PVaR_{1-\alpha}(x^*)$.*

$$
\begin{array}{lll}
\min & Z & \\
\text{s.t.} & \sum_{i=1}^{n} r_i x_i \geq r_0 & \\
& Z^k \geq L_t^k(x);\ t = 1, 2, \cdots, D; k = 1, 2, \cdots, N & \\
& Qy_k + Z \geq Z^k;\ k = 1, 2, \cdots, N & (c) \\
& y_k \in \{0, 1\};\ k = 1, 2, \cdots, N & (d) \\
& \sum_{k=1}^{N} y_k \leq \lfloor \alpha N \rfloor & \\
& \sum_{i=1}^{n} x_i = 1;\ 0 \leq x_i \leq 1 &
\end{array}
\qquad (5.4)
$$

where Q is a big positive number.

Proof

Let (x^*, y^*, Z^*) be the optimal solution of the Model (5.4).

Obviously, the feasible set of Model (5.3) is a subset of Model (5.4), as the Figure 5.1 shows. If we can show that (x^*, y^*, Z^*) is a feasible solution of Model (5.3), then (x^*, y^*, Z^*) is the optimal solution of Model (5.3).

Comparing the constraints of models (5.3) and (5.4), we can see that the differences are constraints (b2)(b3) and (c)(d). Since (x^*, y^*, Z^*) is the optimal solution

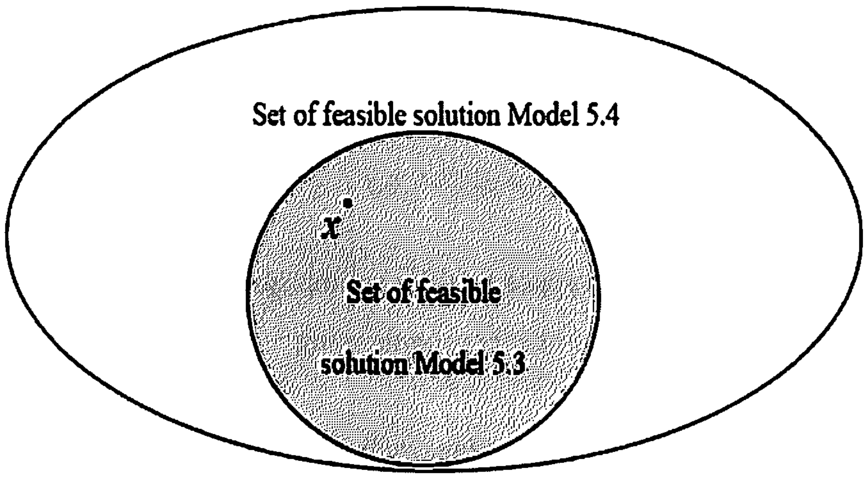

Figure 5.1: Set of feasible solutions

of Model (5.4), constraint (c) implies that y_k^* must be 1 when $Z^* < Z^{k*}$ holds, i.e., (b3) is satisfied at (x^*, y^*, Z^*).

If there exists one j such that $y_j^* = 1$ and $Z^* \geq Z^{j*}$ hold, then we can set $y1$ as follows,

$$y1_k = y_k^*, k = 1, 2, \cdots, N, k \neq j, y1_j = 1.$$

then $(x^*, y1, Z^*)$ is a feasible solution of Model (5.4).

Since the objective value at $(x^*, y1, Z^*)$ is the same as at (x^*, y^*, Z^*), both are optimal solutions of Model (5.4), which contradicts to the assumption that Model (5.4) has an unique optimal solution. Consequently, $y1_k$ must be 0 when $Z^* \geq Z^{k*}$ holds, so (b2) is also satisfied at (x^*, y^*, Z^*).

Hence (x^*, y^*, Z^*) is a feasible solution of Model (5.3), and consequently its optimal solution.

This ends the proof.

That means, we can get an optimal solution of model(5.2) by solving the equivalent model (5.4).

5.2.2 Computation experiments

To illustrate the method proposed in previous section, we selected 3 combinations of five equities from the components of Dow Index, which are listed in Table 5.1 below.

Table 5.1: Three combinations of five equities

	Symbols of equities
Combination 1	BA, CVX, DIS, GE, XOM
Combination 2	INTC, NKE, UNH, V, WMT
Combination 3	AXP, CVX, IBM, PFE, UNH

These combinations are selected by their correlation among the components. Where the correlation in combination 1 is high, that in combination 2 is low, while the correlation in combination 3 is mild. The correlation among equities are calculated using historical price data in 2011, which were obtained from Yahoo!Finance.

To determine N properly, we calculate the PVaR at different confidence levels of these combinations where investment ration on each stock is the same. We calculate the PVaR for every combination 10 times and the average are shown in Table 5.2.

From the Table 5.2, we find that the averages of PVaR estimated from 1000

Table 5.2: Comparison of the averages

Times	100			1000			10000		
$1-\alpha$	95%	90%	85%	95%	90%	85%	95%	90%	85%
Case1	0.14925	0.11513	0.09668	0.14616	0.11424	0.09355	0.14876	0.11547	0.09569
Case2	0.09066	0.06935	0.05786	0.09468	0.07271	0.05957	0.09439	0.07184	0.05835
Case3	0.11337	0.08409	0.06802	0.10882	0.08319	0.06745	0.10734	0.08212	0.06754

scenarios is very close to that estimated from 10000 scenarios. So we set N to 1000. We set r_0 to a value between the maximum return and the minimum return of these equities.

We solve model (5.4) with these combinations. And the results of PVaR and the optimal investment decision are summarized in Table 5.3.

Table 5.3: PVaR and the optimal investment decision

	$1-\alpha$	$PVaR_{1-\alpha}$	x					Computing Time
Case1	97.5%	0.17001	0.2244	0.37339	0.03473	0.05718	0.3103	8209s
	95%	0.14612	0.2159	0.32415	0.03095	0.05875	0.37025	121007s
Case2	97.5%	0.165693	0.0021261	0.014194	0.18759	0.19569	0.6004	10139s
	95%	0.143334	0	0.0067	0.20137	0.19602	0.59591	45298s
Case3	97.5%	0.11821	0.07512	0.06469	0.39466	0.35572	0.10981	27223s
	95%	0.10297	0.13416	0.048448	0.33608	0.39424	0.087079	310314s

These computing experiments show that model (5.4) can be solved in limited time. Meanwhile we know that the solving time is different with different scenarios, and the solving time will be increase, when the number of scenarios increase.

5.3 Return Maximization: model and its solution method

We will build a return maximization model for portfolio selection problems with PVaR as the indicator of market risk, and propose a resolution method for it in this section.

5.3.1 A model and its solution method

We use PVaR as the risk indicator, the return maximization model for portfolio selection problems is given as follows:

$$\begin{array}{ll} max & Return(x) \\ s.t. & PVaR_{1-\alpha}(x) \geq risk_0 \\ & \sum_{i=1}^{n} x_i = 1; 0 \leq x_i \leq 1 \end{array} \tag{5.5}$$

where $x = (x_1, x_2, \cdots, x_n)$ is the decision variable, x_i denotes the percentage of wealth allocated to asset i, $risk_0$ indicates the acceptable risk for investors.

Under Assumption 3 in Chapter 4, future price of stocks during the period can be described as a multi-dimensional geometric Brownian motion, we can use the simulation method to estimate PVaR.

To calculate PVaR for a portfolio consisted of n stocks, we simulate these prices over time span $[0, T]$ N times, producing 252 close daily prices for each stock in one year. Let D be the number of time spots in the maximum investment period $[0, T]$ where stock prices are simulated, the time interval of two adjacent time spots be Δt.

We use the following notations:

S_{it}^{k}: price of stock i at time t in the k^{th} scenario

S_{i0}: purchase price of stock i

r_{it}^k: profit rate of stock i at time t in the k^{th} scenario, $r_{it}^k = \frac{S_{it}^k - S_{i0}}{S_{i0}}$

r_i: expectation of r_{it}^k, $r_i = \frac{1}{ND}\sum_{k=1}^{N}\sum_{t=1}^{D} r_{it}^k$; $i = 1, 2, \cdots, n$

$R_t^k(x)$: profit rate of portfolio at time t in the k^{th} scenario

$L_t^k(x)$: loss rate of portfolio at time t in the k^{th} scenario

$L_T^k(x)$: largest loss rate of portfolio over time span $[0, T]$ in the k^{th} scenario

Then we have the following formulas.

$$R_t^k(x) = \sum_{i=1}^{n} r_{it}^k x_i;\ t = 1, 2, \cdots, D;\ k = 1, 2, \cdots, N.$$

$$L_t^k(x) = -R_t^k(x); \qquad L_T^k(x) = max\{L_t^k(x); t = 1, 2, \cdots, D\},$$

the $\lceil (1-\alpha)N \rceil^{th}$ smallest value in set $\{L_T^k(x); k = 1, 2, \cdots, N\}$ is $PVaR_{1-\alpha}$.

Hence, model (5.5) can be rewritten as follows:

$$\begin{array}{lll}\max & \sum_{i=1}^{n} r_i x_i & \\ \text{s.t.} & L_T^k(x) = max\{L_t^k(x); t = 1, 2, \cdots, D\};\ k = 1, 2, \cdots, N & (e) \\ & \{L_T^k(x); k = 1, 2, \cdots, N\}^{\lceil (1-\alpha)N \rceil} \le risk_0 & (f) \\ & \sum_{i=1}^{n} x_i = 1;\ 0 \le x_i \le 1 & \end{array} \quad (5.6)$$

Because of the collation function and a maximum function in constraint condition, solving model (5.6) directly is difficult. We will suggest to solve this model by changing it into an equivalent model in the follows.

We first prove that the following model is an equivalent model of model (5.6).

$$
\begin{array}{lll}
\max & \sum_{i=1}^{n} r_i x_i & \\
\text{s.t.} & Z^k \geq L_t^k(x); t = 1,2,\cdots,D;\ k = 1,2,\cdots,N & (e1) \\
& Qy_k + risk_0 \geq Z^k;\ k = 1,2,\cdots,N & (f1) \\
& y_k \in \{0,1\};\ k = 1,2,\cdots,N & (f2) \\
& \sum_{k=1}^{N} y_k \leq \lfloor \alpha N \rfloor & (f3) \\
& \sum_{i=1}^{n} x_i = 1;\ 0 \leq x_i \leq 1 &
\end{array}
\qquad (5.7)
$$

where Q is a big positive number, Z^k is an intermediate variable, expressing the largest loss of portfolio over time span $[0, T]$ in the k^{th} scenario, and y_k is an intermediate variable, recording the number of the largest losses bigger than $risk_0$.

Proposition 4 *Models (5.6) and (5.7) are equivalent.*

Proof

We define the following notations.

x^*: any one feasible solution of model (5.6).

x': any one feasible solution of model (5.7).

(1)We prove that x^* is a feasible solution of model(5.7).

Because x^* is a feasible solution of model (5.6), $L_T^k(x^*) = max\{L_t^k(x^*); t = 1,2,\cdots,D\}$, let $Z^k = L_T^k(x^*)$, x^* meet all constraints in model (5.7). Hence, x^* is a feasible solution of model (5.7) also.

(2)We prove that x' is a feasible solution of model (5.6).

Since x' is a feasible solution of model (5.7), from the constraint (e1) we know $Z^k \geq L_T^k(x')$, and constraints (f1)~ (f3) imply that y_k^* must be 1 when $risk_0 < Z^k$ holds, the numbers of $Z^k > risk_0$ is not bigger than $\lfloor \alpha N \rfloor$. It means that the

$\lceil (1-\alpha)N \rceil^{th}$ smallest value in the $\{L_T^k(x'); k = 1, 2, \cdots, N\}$ is equal or less than $risk_0$. Hence, all constraints are satisfied with x' in model (5.6). Hence, x' is the feasible solution of model (5.6) also.

Model (5.6) and model (5.7) has the same set of feasible solution.

This ends the proof.

That means we can get an optimal solution of model (5.6) by solving the equivalent model (5.7).

5.3.2 Computation experiments

We use the same data and the same combinations in previous section to solve model 5.7.

Case one: BA, CVX, DIS, GE and XOM.

Case two: INTC, NKE, UNH, V and WMT.

Case three: AXP, CVX, IBM, PFE and UNH.

Investment period: one year, from 2013/8/1.

N: 1000.

$risk_0$: 0.20.

The return and the optimal investment decision are summarized in Table 5.4. These computing experiments show that model (5.7) can be solved in limited time.

Table 5.4: Return and the optimal investment decision

	$1-\alpha$	*return*	x					Computing Time
Case1	97.5%	0.107783	0.17033	0.52261	0.001054	0.000326	0.30568	27
	95%	0.110949	0.1915	0.65807	0	0	0.15043	62
	90%	0.114031	0.091324	0.81225	0.000653	0.00032	0.095453	83
Case2	97.5%	0.209051	0	0	0.12944	0.16836	0.7022	26
	95%	0.21555	0	0	0.09086	0.12854	0.7806	40
	90%	0.225649	0	0.00885	0.060062	0.024658	0.90643	18
Case3	97.5%	0.2419996	0	0.00192	0.32066	0.67742	0	3
	95%	0.246214	0	0	0.1766	0.8234	0	9
	90%	0.250147	0	0	0.03147	0.96853	0	8

5.4 Summary

To solve investment decision models with PVaR included, we considered portfolio selection problems with PVaR as the indicator of market risk, and PVaR is estimated using Monte Carlo simulation. The risk minimization model turns out to be a complicated nonlinear programming, we suggested to solve the model by changing it to an equivalent mixed programming model, and illustrated this method with numerical examples. We did computing experiments with data from financial market, and showed that this method is valid for models with a small size, but computing load increased rapidly when the model size grows.

We also considered the return maximization model for portfolio selection problem. Since, the model is a complicated nonlinear programming, we suggested to solve it by changing it to a mixed programming model, and illustrated this method with numerical examples. We did computing experiments with data from financial market, and showed that this method is valid for models with a small size.

Chapter 6

Summary and Future Research

6.1 Summary

This study proposed and formulated the notion of financial risk over a period of time. We extended VaR to Period Value at Risk as an indicator of risk for a given time span. PVaR is a brand new indicator for market risk over a period of time, we proposed methods for computing PVaR of an investment and solving investment decision models with PVaR included, these results are expected to facilitate the use of PVaR in making financial investment decisions.

To calculate PVaR of an investment, we derived an analytic method for computing PVaR under the conditions that there was only one risk factor which can be modeled as a geometric Brownian motion. For more general cases where multiple risk factors are involved, we proposed to use Monte Carlo simulation to estimate the PVaR of an investment. We illustrated this method for a portfolio composed of correlated financial instruments, and the prices of component financial instruments can be expressed with geometric Brownian motions, whose parameters can be identified with historical price data. Our computing experiments showed that the simulation method provides a usable way for estimating PVaR, making PVaR operational in investment practice.

To solve investment decision models with PVaR included, we considered portfolio selection problems with PVaR as the indicator of market risk, and PVaR was estimated using Monte Carlo simulation. The risk minimization model and return maximization model turn out to be a complicated nonlinear programming, we suggested to solve these models by changing them to an equivalent mixed programming model, and illustrated these methods with numerical examples. We did computing experiment with data from financial market, and showed that this method is valid for models with a small size, but computing load increased rapidly when the model size grows.

6.2 Future Research

The future research topics include more effect methods for solving investment models with PVaR as the risk indicator and development of financial products with PVaR as risk indicator.

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Appendix A

Proof of Lemma 1 in Section 3.2

Define

$$M(t) = exp[-aB(t) - \frac{1}{2}a^2t], 0 \leq t \leq T.$$

Because $exp[\frac{1}{2}a^2T] < \infty$ holds and that $B(t)$ is a standard Brownian motion under probability measure P, we know from the Girsanov theorem that $X(t) = at + B(t)$ is a standard Brownian motion under probability measure Q, where Q is defined by

$$dQ = M(T)dP.$$

Define set $A = \{(X_T, X(T)) | X_T \leq x, X(T) \leq y\}$ and variable

$$1_A(X_T, X(T)) = \begin{cases} 1, & (X_T, X(T)) \in A \\ 0, & (X_T, X(T)) \notin A, \end{cases}$$

then we have

$$\begin{aligned} F(x,y) &= P(X_T \leq x, X(T) \leq y) = E_P[1_A(X_T, X(T))] \\ &= E_Q[M^{-1}(T)1_A(X_T, X(T))] \\ &= E_Q[exp(aB(T) + \tfrac{1}{2}a^2T)1_A(X_T, X(T))] \\ &= E_Q[exp(aX(T) - \tfrac{1}{2}a^2T)1_A(X_T, X(T))]. \end{aligned}$$

Because $X(t)$ is a standard Brownian motion under probability measure Q, the joint distribution density function of $X(T)$ and X_T is given as follows (Karatzas and Shreve, 1991: p. 95).

$$f_{(X_T,X(T))}(\theta,z)=\begin{cases}\sqrt{\frac{2}{\pi T^3}}(2\theta-z)exp(\frac{-(2\theta-z)^2}{2T}), & \theta\geq z,\theta>0\\ 0, & otherwise.\end{cases}$$

Hence, we have

$$\begin{aligned}F(x,y) &= P(X_T\leq x, X(T)\leq y)=\int_0^x\int_{-\infty}^y exp(az-\tfrac{1}{2}a^2T)f_{(X_T,X(T))}(\theta,z)d\theta dz\\
&= \int_0^x\int_{-\infty}^y exp(az-\tfrac{1}{2}a^2T)\sqrt{\tfrac{2}{\pi T^3}}(2\theta-z)exp[-(2\theta-z)^2/2T]d\theta dz\\
&= \int_{-\infty}^y exp(az-\tfrac{1}{2}a^2T)\sqrt{\tfrac{2}{\pi T^3}}\int_0^x(2\theta-z)exp[-(2\theta-z)^2/2T]\tfrac{-T}{2(2\theta-z)}d(-(2\theta-z)^2/2T)dz\\
&= \int_{-\infty}^y exp(az-\tfrac{1}{2}a^2T)\sqrt{\tfrac{1}{2\pi T}}\int_{-(2x-z)^2/2T}^{-z^2/2T}exp(\theta)d\theta dz\\
&= \int_{-\infty}^y exp[az-\tfrac{1}{2}a^2T]\sqrt{\tfrac{1}{2\pi T}}[exp(-z^2/2T)-exp(-(2x-z)^2/2T)]dz\\
&= \sqrt{\tfrac{1}{2\pi T}}\int_{-\infty}^y exp(-z^2/2T+az-\tfrac{1}{2}a^2T)-exp(az-\tfrac{1}{2}a^2T-(2x-z)^2/2T)dz\\
&= \sqrt{\tfrac{1}{2\pi T}}\int_{-\infty}^y exp(\tfrac{-z^2+2Taz-(aT)^2}{2T})-exp(\tfrac{az(2T)-(aT)^2-(2x-z)^2}{2T})dz\\
&= \sqrt{\tfrac{1}{2\pi T}}\int_{-\infty}^y exp(\tfrac{-z^2+2Taz-(aT)^2}{2T})-exp(\tfrac{2x(2aT)+(z-2x)(2aT)-(aT)^2-(2x-z)^2}{2T})dz\\
&= \sqrt{\tfrac{1}{2\pi T}}\int_{-\infty}^y exp(\tfrac{-(z-aT)^2}{2T})-exp(2xa)exp(\tfrac{-(z-2x-aT)^2}{2T})dz\\
&= \Phi(\tfrac{y-aT}{\sqrt{T}})-exp(2xa)\Phi(\tfrac{y-2x-aT}{\sqrt{T}}).\end{aligned}$$

End of proof.

Appendix B

Proof of Proposition 1 in Section 3.2

The loss rate indicated in formula (3.4) is a Brownian motion with drift. According to the symmetry of the standard Brownian motion $B(t)$, the loss-rate function can be written as

$$L(\omega(t)) = -(\mu - \frac{\sigma^2}{2})t + \sigma B(t), \tag{B.1}$$

which is also a Brownian motion with drift.

Let $L_T = \max_{0 \le t \le T} L(\omega(t))$. We know from Lemma 2 that the joint distribution function of L_T and $L(\omega(T))$ is as follows:

$$\begin{aligned} F(x,y) = & \ P(L_T \le x, L(\omega(T)) \le y) \\ = & \ \Phi(\frac{y-(\frac{\sigma^2}{2}-\mu)T}{\sigma\sqrt{T}}) - exp(\frac{2x(\frac{\sigma^2}{2}-\mu)}{\sigma^2})\Phi(\frac{y-2x-(\frac{\sigma^2}{2}-\mu)T}{\sigma\sqrt{T}}). \end{aligned} \tag{B.2}$$

Denote the distribution function of L_T by $F(x)$. Because $P(L_T \le x) = P(L_T \le x, L(\omega(T)) \le x) + P(L_T \le x, L(\omega(T)) \ge x)$, and $P(L_T \le x, L(\omega(T)) \ge x) = 0$ hold, we have $F(x) = P(L_T \le x) = P(L_T \le x, L(\omega(T)) \le x)$.

Replacing y in formula (B.2) with x, we can get the distribution function of L_T

as follows:

$$\begin{aligned} F(x) = P(L_T \leq x) &= \Phi(\frac{x-(\frac{\sigma^2}{2}-\mu)T}{\sigma\sqrt{T}}) - exp(\frac{2x(\frac{\sigma^2}{2}-\mu)}{\sigma^2})\Phi(\frac{-x-(\frac{\sigma^2}{2}-\mu)T}{\sigma\sqrt{T}}) \\ &= \Phi(\frac{x+(\mu-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}) - exp(\frac{-2x(\mu-\frac{\sigma^2}{2})}{\sigma^2})\Phi(\frac{-x+(\mu-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}). \end{aligned}$$

End of proof.

Appendix C

Correlation Coefficients

	AXP	BA	CAT	CSCO	CVX	DD	DIS	GE	GS	HD	IBM	INTC	JNJ	JPM	KO
AXP	1	0.746773	0.736867	0.552909	0.703501	0.760166	0.7428	0.75676	0.760933	0.646446	0.635324	0.5882	0.67467	0.803596	0.650168
BA	0.746773	1	0.770767	0.54215	0.721736	0.760574	0.782171	0.753164	0.665897	0.661313	0.701338	0.633674	0.668016	0.696505	0.68726
CAT	0.736867	0.770767	1	0.572921	0.798452	0.85107	0.746241	0.724714	0.676904	0.621497	0.664953	0.634462	0.677017	0.726823	0.621139
CSCO	0.552909	0.54215	0.572921	1	0.578163	0.583303	0.554407	0.575986	0.582124	0.500221	0.525404	0.544479	0.562883	0.582626	0.496297
CVX	0.703501	0.721736	0.798452	0.578163	1	0.803719	0.713484	0.742443	0.648253	0.647397	0.625367	0.580799	0.707574	0.702501	0.683762
DD	0.760166	0.760574	0.85107	0.583303	0.803719	1	0.773421	0.769411	0.714997	0.660296	0.67401	0.627005	0.714763	0.767573	0.696479
DIS	0.7428	0.782171	0.746241	0.554407	0.713484	0.773421	1	0.768488	0.699911	0.642788	0.678413	0.607358	0.654975	0.738012	0.656504
GE	0.75676	0.753164	0.724714	0.575986	0.742443	0.769411	0.768488	1	0.733066	0.650769	0.657613	0.625478	0.684618	0.796249	0.676862
GS	0.760933	0.665897	0.676904	0.582124	0.648253	0.714997	0.699911	0.733066	1	0.565415	0.541254	0.543449	0.609322	0.889565	0.557194
HD	0.646446	0.661313	0.621497	0.500221	0.647397	0.660296	0.642788	0.650769	0.565415	1	0.575523	0.571765	0.589996	0.601924	0.631951
IBM	0.635324	0.701338	0.664953	0.525404	0.625367	0.67401	0.678413	0.657613	0.541254	0.575523	1	0.633732	0.602186	0.602748	0.694674
INTC	0.5882	0.633674	0.634462	0.544479	0.580799	0.627005	0.607358	0.625478	0.543449	0.571765	0.633732	1	0.571545	0.570059	0.57449
JNJ	0.67467	0.668016	0.677017	0.562883	0.707574	0.714763	0.654975	0.684618	0.609322	0.589996	0.602186	0.571545	1	0.650098	0.661545
JPM	0.803596	0.696505	0.726823	0.582626	0.702501	0.767573	0.738012	0.796249	0.889565	0.601924	0.602748	0.570059	0.650098	1	0.585905
KO	0.650168	0.68726	0.621139	0.496297	0.683762	0.696479	0.656504	0.676862	0.557194	0.631951	0.694674	0.57449	0.661545	0.585905	1
MCD	0.619136	0.608088	0.557643	0.469231	0.584311	0.582963	0.569629	0.556364	0.487744	0.60915	0.568979	0.492169	0.591587	0.520923	0.658909
MMM	0.760896	0.760528	0.80589	0.598948	0.771571	0.799806	0.775277	0.797939	0.713581	0.654052	0.676061	0.650945	0.736881	0.764958	0.684644
MRK	0.674902	0.683704	0.631019	0.531228	0.669343	0.670933	0.689897	0.688775	0.57415	0.574065	0.62734	0.547428	0.720239	0.632555	0.684302
MSFT	0.670594	0.675077	0.672689	0.63041	0.676658	0.703819	0.684609	0.681162	0.616059	0.631761	0.682591	0.662063	0.606554	0.666005	0.630941
NKE	0.562248	0.593447	0.612767	0.507456	0.582537	0.596866	0.571224	0.575607	0.471536	0.569094	0.548015	0.543645	0.540051	0.522557	0.598459
PFE	0.660881	0.661427	0.653362	0.525307	0.695621	0.690313	0.650876	0.70651	0.574938	0.639461	0.623725	0.550615	0.710511	0.628282	0.629679
PG	0.638194	0.634056	0.56915	0.501971	0.628252	0.617922	0.611371	0.614986	0.539721	0.556306	0.610044	0.551303	0.66454	0.594202	0.66062
T	0.659371	0.64805	0.612739	0.455164	0.660072	0.687891	0.670087	0.682892	0.585667	0.59747	0.677443	0.545305	0.621279	0.662839	0.637842
TRV	0.690248	0.693691	0.67962	0.55146	0.706989	0.710568	0.710754	0.678179	0.680442	0.625296	0.590152	0.608519	0.625399	0.72771	0.674965
UNH	0.646401	0.644679	0.642641	0.503662	0.628392	0.641203	0.65188	0.634174	0.549973	0.612165	0.61126	0.54845	0.620311	0.594262	0.618005
UTX	0.770378	0.81834	0.844137	0.611905	0.779919	0.815454	0.786531	0.776791	0.68001	0.673682	0.74819	0.692028	0.725406	0.719353	0.710808
V	0.666821	0.572939	0.586868	0.484556	0.549734	0.61114	0.561214	0.57391	0.572493	0.545395	0.531031	0.526223	0.569755	0.592017	0.522051
VZ	0.612584	0.576261	0.590119	0.473472	0.627938	0.631136	0.617578	0.685984	0.519111	0.568029	0.586027	0.509454	0.635528	0.608032	0.593831
WMT	0.575817	0.58767	0.538195	0.516874	0.556602	0.566875	0.581024	0.555804	0.505115	0.692925	0.469173	0.489746	0.57458	0.553057	0.569757
XOM	0.699802	0.734805	0.753005	0.58252	[illegible]	0.774908	0.720534	0.762472	0.653262	0.639052	0.675966	0.610824	0.724311	0.713232	0.709293

Figure C.1: Correlation coefficients of 30 stocks

MCD	MMM	MRK	MSFT	NKE	PFE	PG	T	TRV	UNH	UTX	V	VZ	WMT	XOM	
0.619136	0.760896	0.674902	0.670594	0.562248	0.660881	0.638194	0.659371	0.690248	0.646401	0.770378	0.666821	0.612584	0.575817	0.699802	AXP
0.608088	0.760528	0.683704	0.675077	0.593447	0.661427	0.634056	0.64805	0.693691	0.644679	0.81834	0.572939	0.576261	0.58767	0.734805	BA
0.557643	0.80589	0.631019	0.672689	0.612767	0.653362	0.56915	0.612739	0.67962	0.642641	0.844137	0.586868	0.590119	0.538195	0.753005	CAT
0.469231	0.598948	0.531228	0.63041	0.507456	0.525307	0.501971	0.455164	0.55146	0.503662	0.611905	0.484556	0.473472	0.516874	0.58252	CSCO
0.584311	0.771571	0.669343	0.676658	0.582537	0.695621	0.628252	0.660072	0.706989	0.628392	0.779919	0.549734	0.627938	0.556602	0.905136	CVX
0.582963	0.799806	0.670933	0.703819	0.596866	0.690313	0.617922	0.687891	0.710568	0.641203	0.815454	0.61114	0.631136	0.566875	0.774908	DD
0.569629	0.775277	0.689897	0.684609	0.571224	0.650876	0.611371	0.670087	0.710754	0.65188	0.786531	0.561214	0.617578	0.581024	0.720534	DIS
0.556364	0.797939	0.688775	0.681162	0.575607	0.70651	0.614986	0.682892	0.678179	0.634174	0.776791	0.57391	0.685984	0.555804	0.762472	GE
0.487744	0.713581	0.57415	0.616059	0.471536	0.574938	0.539721	0.585667	0.680442	0.549973	0.68001	0.572493	0.519111	0.505115	0.653262	GS
0.60915	0.654052	0.574065	0.631761	0.569094	0.639461	0.556306	0.59747	0.625296	0.612165	0.673682	0.545395	0.568029	0.692925	0.639052	HD
0.568979	0.676061	0.62734	0.682591	0.548015	0.623725	0.610044	0.677443	0.590152	0.61126	0.74819	0.531031	0.586027	0.489173	0.675966	IBM
0.492169	0.650945	0.547428	0.662063	0.543645	0.550615	0.551303	0.545305	0.608519	0.54845	0.692028	0.526223	0.509454	0.489746	0.610824	INTC
0.591587	0.736881	0.720239	0.606554	0.540051	0.710511	0.66454	0.621279	0.625399	0.620311	0.725406	0.569755	0.635528	0.57458	0.724311	JNJ
0.520923	0.764958	0.632555	0.666005	0.522557	0.628282	0.594202	0.662839	0.72771	0.594262	0.719353	0.592017	0.608032	0.553057	0.713232	JPM
0.658909	0.684644	0.684302	0.630941	0.598459	0.629679	0.66062	0.637842	0.674965	0.618005	0.710808	0.522051	0.593831	0.569757	0.709293	KO
1	0.639521	0.610423	0.569038	0.631974	0.599531	0.566233	0.555418	0.598971	0.533972	0.62505	0.510664	0.522743	0.571956	0.566918	MCD
0.639521	1	0.68864	0.705497	0.632743	0.687126	0.655894	0.646032	0.739274	0.610992	0.848424	0.606957	0.636127	0.610972	0.748141	MMM
0.610423	0.68864	1	0.620473	0.529457	0.722495	0.645911	0.641992	0.654588	0.61164	0.694634	0.571759	0.620664	0.571427	0.699165	MRK
0.569038	0.705497	0.620473	1	0.593603	0.620668	0.579561	0.609523	0.600323	0.562417	0.723456	0.566461	0.563981	0.579684	0.703494	MSFT
0.631974	0.632743	0.529457	0.593603	1	0.551402	0.502822	0.510483	0.562364	0.545494	0.612164	0.524174	0.478182	0.488725	0.592289	NKE
0.599531	0.687126	0.722495	0.620668	0.551402	1	0.576964	0.598003	0.642441	0.590032	0.675389	0.557797	0.628322	0.552283	0.715801	PFE
0.566233	0.655894	0.645911	0.579561	0.502822	0.576964	1	0.669276	0.607009	0.492556	0.677481	0.455405	0.592598	0.618861	0.666614	PG
0.555418	0.646032	0.641992	0.609523	0.510483	0.598003	0.669276	1	0.657865	0.605463	0.649827	0.547401	0.757434	0.531049	0.692612	T
0.598971	0.739274	0.654588	0.600323	0.562364	0.642441	0.607009	0.657865	1	0.66542	0.683107	0.53351	0.589591	0.619882	0.706214	TRV
0.533972	0.610992	0.61164	0.562417	0.545494	0.590032	0.492556	0.605463	0.66542	1	0.660999	0.472764	0.486722	0.498452	0.654665	UNH
0.62505	0.848424	0.694634	0.723456	0.612164	0.675389	0.677481	0.649827	0.683107	0.660999	1	0.610945	0.60549	0.601957	0.776531	UTX
0.510664	0.606957	0.571759	0.566461	0.524174	0.557797	0.455405	0.547401	0.53351	0.472764	0.610945	1	0.46822	0.421717	0.571699	V
0.522743	0.636127	0.620664	0.563981	0.478182	0.628322	0.592598	0.757434	0.589591	0.486722	0.60549	0.46822	1	0.549575	0.618554	VZ
0.571956	0.610972	0.571427	0.579684	0.488725	0.552283	0.618861	0.531049	0.619882	0.498452	0.601957	0.421717	0.549575	1	0.550944	WMT
0.566918	0.748141	0.699165	0.703494	0.592289	0.715801	0.666614	0.692612	0.706214	0.654665	0.776531	0.571699	0.618554	0.550944	1	XOM

Figure C.2: Correlation coefficients of 30 stocks

Date	AXP	BA	CAT	CSCO	CVX	DD	DIS	GE	GS	HD	IBM	INTC	JNJ	JPM	KO	MCD	MMM	MRK	MSFT	NKE	PFE	PG	T	TRV	UNH	UTX	V	VZ	WMT	XOM
12/30	45.9	70.5	87	17.2	100	43.1	36.9	16.9	88.1	40.5	178	22.8	61.8	31.4	33.3	94.9	78.2	35.2	24.7	47	20.3	63.1	27.3	56.5	49	70.1	100	36.6	57	81
12/29	46.3	71.2	87	17.4	102	43.2	37.1	17.1	88.7	40.5	181	23	62.1	31.6	33.4	95.4	78.6	35.2	24.8	47.5	20.4	63.3	27.2	57	50	70.8	102	36.5	57	81.5
12/28	45.7	70.4	85.9	17.3	100	42.8	36.7	16.8	87.8	40	178	22.7	61.7	30.8	33.1	94.2	77.7	35	24.6	47	20.2	62.9	27	56.2	49	70.1	99.4	36.3	57	80.5
12/27	46.4	71.4	88	17.7	102	43.6	37	17	89.5	40.7	179	23	62.2	31.2	33.3	95.1	78.7	35.2	24.8	47.7	20.4	63.2	27.1	56.8	50	71	101	36.5	57	81.5
12/23	46.7	71.1	88.6	17.6	102	43.5	37.1	17.2	91.4	40.6	179	22.9	62.2	31.7	33.3	94.8	78.7	35.4	24.8	47.3	20.5	63	26.9	56.7	50	71.1	101	36.5	57	81.5
12/22	46.4	71.4	88.2	17.3	100	42.9	36.4	17	92	40.4	177	22.5	61.4	31.6	32.9	93.3	77.5	35.1	24.6	46.2	20.3	62.6	26.8	56.5	49	70.5	99.2	35.9	57	80.6
12/21	46.2	70.7	88	17.1	99.6	42.6	35.7	16.5	89.6	40.5	176	22.2	61.2	30.5	33.1	93.9	76.6	34.9	24.5	47	20.4	62.4	26.4	56.4	49	70.6	99.4	35.8	57	79.5
12/20	46.7	69.6	88.1	17.5	97.9	42.4	35.6	16.2	88.6	40.4	182	22.4	60.8	30.4	32.6	93.5	76.9	34.6	24.8	45.7	20.1	62.2	26.3	55.5	49	71.3	100	35.8	57	78.4
12/19	44.8	67.4	83.8	16.9	94.2	40.7	34.2	15.8	85.4	38.7	177	21.7	59.9	29	32.1	92	74.5	34.1	24.3	45.5	19.8	61.4	25.9	54.2	48	68.8	97.3	35.3	55	76
12/16	45.7	68.2	83.8	17.1	95.3	41.4	34.8	15.9	87.8	38.9	178	21.8	60.6	30.1	32.1	92.2	75.5	33.8	24.7	45.7	19.7	61.6	26	54.7	48	69.4	96.1	35.4	56	76.6
12/15	45.2	67.9	84.3	17.2	94.1	41.2	34.7	15.7	89.5	38	182	21.9	60.3	30	31.8	92.9	75.5	34	24.3	45.9	19.8	61.5	26	54.3	47	70.5	94.5	35.1	56	76.5
12/14	45.8	67.2	83.6	17.1	94.9	40.7	34.6	15.5	90.8	37.7	183	21.9	59.5	29.8	31.5	92.3	75.1	33.2	24.3	46	19.6	60.9	26	53.4	47	70.1	95.1	34.9	55	75.9
12/13	46.2	68.1	87.4	17.6	97.9	41	35.2	15.4	92.6	38.1	185	22.1	59.7	29.6	31.7	92.7	76	32.7	24.5	46.3	19.5	61.2	26.2	53.4	47	71.1	94.6	34.9	55	77
12/12	46.6	68.1	89.6	17.7	97.3	41.4	35.5	15.4	95.5	38.6	186	22.5	59.9	30.3	31.8	93.2	77.1	32.7	24.3	47.3	19.1	60.8	26.2	53.3	47	71.4	95.6	35	56	76.5
12/9	47.5	69.1	92.2	18	98.5	42.4	35.4	15.7	98.8	38.8	189	23.5	60.8	31.3	32.2	92.7	78.7	32.9	24.4	47.6	19.3	61.4	26.2	53.5	48	73.1	95.9	35.1	56	77.8
12/8	46.6	67.4	89.3	17.7	96.6	43.8	34.8	15.3	97.3	38.5	186	23.2	60.1	30.4	31.8	91.7	77	32.5	24.2	46.4	19	61	26	53	47	71.3	94.5	34.5	56	76.3
12/7	47.9	67.8	91.2	18.1	98.7	45.3	35.9	15.7	102	39.2	188	24.1	60.7	32.1	31.8	91.3	78.9	32.9	24.4	46.8	19.2	61.7	26.5	53.4	48	73.1	95.8	35	56	77.5
12/6	47.3	68.1	92.2	17.8	98.6	45.2	35.7	15.6	98.5	38.8	187	23.8	59.8	31.4	31.7	90.8	78.6	32.7	24.4	46.9	19	61.3	26.3	52.5	47	73.1	94.1	35	56	77.3
12/5	47.5	68.3	93	17.9	97.1	45.2	35.8	15.3	97.2	38.8	185	23.5	59.7	31.7	31.6	90.2	77.5	32.5	24.4	47.1	18.6	61.3	26.3	51.8	47	73.5	94.8	34.7	56	76.9
12/2	47	68.5	92.5	17.7	96	44.3	35.5	15	94.7	38.5	184	23.1	59.8	30.5	31.6	90.5	76.3	32.7	24	46.9	18.7	61.1	26.1	51.4	47	73.4	95.9	34.5	55	76.3
12/1	46.5	68.2	93	17.7	96.2	44.6	34.9	14.9	92	37.9	184	23.4	60.7	28.8	31.8	90.4	76.9	32.9	24	46.4	18.8	60.6	26	52.2	47	73.4	95.4	34.5	56	76.3
11/30	46.8	66	94	17.8	97.1	45	34.7	14.9	93.4	37.8	182	23.4	61	29.3	32	90.4	77.6	33	24.3	46.7	18.8	61.1	26.1	53.3	47	73.4	95.7	34.4	56	76.9
11/29	44.4	62.7	87	16.8	92	42.5	32.9	14	86.5	37.5	175	22.1	59.2	27	31.5	88.4	73.9	31.8	23.6	46.1	18.2	59.1	25.3	51.3	45	70	91.7	33.4	55	73.5
11/28	44.8	62.5	87.9	17.2	90.5	42.4	33	13.8	88.1	35.7	177	22	58.8	27.5	30.8	88	74.4	31.6	23.7	46.1	17.9	58.9	25.2	51.4	44	70.1	92.2	33.1	55	72.5
11/25	43.8	60.3	83.3	16.7	87.2	41.3	32.5	13.7	86.1	34.9	172	21.3	57.7	26.9	30.6	86.5	72.9	30.6	23.1	43.9	17.3	57.7	24.7	50.6	42	68.1	87.8	32.3	54	70.6
11/23	43.9	59.9	84.3	16.6	88.5	41.5	32.4	13.8	85.3	34.9	173	21.3	57.9	26.8	30.7	86.3	72.3	30.6	23.3	44.2	17.3	57.7	24.9	50	42	68	88.1	32.3	54	71.3
11/22	44.8	61.8	86.5	17.1	91.1	42.8	33	14	86.8	35.5	176	21.8	58.7	27.8	31.2	87	74.5	31.2	23.6	44.5	17.7	58.3	25.3	51.1	43	70	89.6	33	54	72.7
11/21	44.9	63	87.5	17.1	90.3	42.9	33.3	14.3	88.6	35.4	176	22.1	58.8	28.3	31.2	86.7	74.5	31.5	23.8	44.2	17.8	58.3	25.6	51.9	43	71	90	32.9	54	73.5
11/18	45.7	64.8	90.2	17.5	92.4	43.7	34.5	14.6	89.2	36.2	180	22.8	59.6	28.9	31.9	87.1	76.5	32.2	24.1	45.1	18.3	59.8	25.8	53.3	43	73	89.6	33.3	54	74.5
11/17	45.5	63.5	90.1	17.6	94.5	43.4	34.1	14.6	89.6	36	180	22.8	59.7	28.8	31.5	86.7	76.4	32.1	24.3	44.6	18.3	59.5	25.8	52.8	43	72.5	90.2	33.5	54	74.4
11/16	46.9	63.8	92	17.9	95.4	44.4	34.4	14.9	92.8	36.2	181	23.4	60.1	29.7	31.7	87.5	77.8	32.4	24.8	45.5	18.4	59.8	26	53.4	44	74.2	92.1	33.4	54	74.8
11/15	48.6	65.3	93.3	18.2	96.8	45.5	35.3	15.1	96.8	36.4	183	23.8	60.7	30.9	32.1	88.7	77.8	33	25.4	46.6	18.7	60.1	26.4	54.3	45	75.6	94.1	34	55	75.6
11/14	48.2	65.3	92.7	18	99.5	45.5	35	15.1	96.4	36.6	182	23.1	60.6	30.8	32	88.3	77.8	32.9	25.3	46.5	18.6	59.6	26.3	54.5	46	75.3	92.3	33.8	56	75.5
11/11	49.1	64.3	92.4	18.1	100	45.7	35.6	15.2	98.7	36.4	182	23.3	60.9	31.4	32.2	89	78.2	33.2	25.4	46.7	18.8	60.4	26.5	55.4	46	76.1	93.7	34.2	56	76.2
11/10	47.8	62.3	88.6	17.7	98.9	44.8	33.6	15	96.6	35.6	178	22.6	59.9	30.9	31.9	87.6	76.3	32.3	24.8	45.6	18.6	59.8	26.3	54.4	45	73.8	91.6	34.1	55	75.2
11/9	47.9	62	88	16.8	97.7	44.5	32.7	14.8	96.7	35.5	177	22.4	59.3	30.7	31.7	87	75.1	31.2	24.7	44.9	18.2	59.3	26.1	54	44	72.8	91.7	33.7	55	74
11/8	50	64	92.1	17.4	102	46.5	34.2	15.4	105	36.3	182	23.2	60.6	33.1	32.5	88.8	77.8	31.8	25.6	46.4	18.9	60.7	26.6	56	45	75.1	93.1	34.2	56	76.2
11/7	49.9	63.3	91.3	17.2	101	45.8	34.1	15.3	102	35.7	181	22.8	59.8	32.3	32.2	88.9	75.7	31.7	25.3	45.6	18.7	60	26.6	55.6	44	74.6	91.5	34.1	55	75.4
11/4	49.8	62.8	92	17.2	99.7	45.5	33.7	15.3	102	34.8	180	22.3	59.9	32.1	32	88.1	75.4	31.4	24.8	45.9	18.3	59.6	26.3	55.1	44	74.1	91.2	33.9	55	74.6
11/3	50.3	63.2	92.2	17.3	99.5	45.6	33.9	15.6	104	34.8	181	22.7	60.1	32.5	32.5	87.3	75.7	31.9	25	46.5	18.5	59.8	26.6	55.5	45	74.7	91.7	34.2	55	74.9
11/2	49	61.5	90.2	17	98	44.7	33	15.2	103	34.4	178	22.2	59.4	31.8	32	86.9	74.2	31.5	24.6	46.1	18.2	59.4	26.2	54.7	45	73.3	90.1	33.6	54	73.5
11/1	47.7	60.3	88	16.8	95.7	43.8	32.6	15	100	34	175	22.2	59.2	30.9	31.7	86.3	72.7	31.4	24.5	45.6	18	59.3	25.9	53.5	45	71.6	88.6	33.3	54	72.2

Figure C.3: The historical stock prices of stocks during 2011

Date	AXP	BA	CAT	CSCO	CVX	DD	DIS	GE	GS	HD	IBM	INTC	JNJ	JPM	KO	MCD	MMM	MRK	MSFT	NKE	PFE	PG	T	TRV	UNH	UTX	V	VZ	WMT	XOM
10/31	49.3	62.8	90.7	17.7	98.4	44.9	33.8	15.6	106	34.2	178	22.8	60.1	32.8	32.3	87.2	75.1	31.8	25.1	46.8	17.9	60.5	26.4	55.3	47	74.3	91.8	33.7	54	74.2
10/28	50.7	65.1	95	17.7	103	46.1	35.1	16.1	112	34.5	181	23.2	61.3	34.7	32.6	87.6	77	32.4	25.5	47	18.4	61.2	26.8	56.4	47	75.7	93.6	34.3	54	77.4
10/27	50.7	64.4	92.5	17.6	102	45.5	35.1	16.2	113	35.6	180	23.4	61.2	35	32.4	87.8	77.4	31.7	25.7	46.6	18.4	61.7	26.6	56.3	48	75.5	92.9	34.4	55	77.8
10/26	49.1	63.6	88	16.8	100	43.1	34	15.3	103	34.9	176	23	60.2	32.3	31.9	86.2	73.2	30.9	25.1	45.8	17.9	61.4	25.9	54.5	47	72.5	90.6	33.6	55	77
10/25	48.1	60.8	86.4	16.8	97.9	42	33.4	15.2	97.5	34.5	174	22.9	59.5	31.6	31.6	86.2	73.2	30.4	25.3	45.5	17.5	61	25.6	53.4	47	71.7	89.3	33	54	75.5
10/24	48.6	61.8	88.2	16.7	99.6	43.1	34.3	15.4	101	35.1	176	22.9	60.4	32.7	32.1	86.4	78.1	30.9	25.7	46.2	18	61.8	26.1	54.6	48	73	92.7	33.7	54	76.2
10/21	47.2	61.7	84	16.6	98.9	42.2	34.1	15.3	99.1	35.2	175	22.3	59.6	31.6	32.2	86.7	76.5	30.8	25.6	45.8	17.7	62.7	26.3	54.4	47	72	92	34.2	54	76.1
10/20	45	59.7	80.9	16.4	96.9	41.2	32.8	15.6	97.9	34.2	171	22	58.4	31.3	31.7	83.6	74.8	30.3	25.5	44.6	17.4	61.5	26.2	51.7	45	70.7	89.9	33.9	54	74.8
10/19	44.9	60.3	79.8	16.3	95.8	40.9	32.6	15.4	97.8	33.8	171	22.5	58.5	30.5	31.7	84.2	73.8	30	25.6	44.3	17.5	61.2	26.2	51.6	45	69.8	88.7	33.8	54	74.5
10/18	45.5	60.6	80.9	16.7	96.5	42	32.9	15.6	99.2	34.4	173	21.8	60.2	31.1	31.5	84.2	74.5	30.3	25.8	44.4	17.6	60.9	26.4	48.8	44	70.6	92.4	34	53	75
10/17	43.9	59	77.9	16.4	92.4	40.9	32.3	15.2	94	33.1	180	21.7	59.6	29.3	31.7	83.3	72.1	29.8	25.5	44.2	17.4	60.3	26.2	47	45	68.7	89.8	33.7	52	73.6
10/14	44.9	61	80.3	16.7	94.1	42.1	33.4	15.5	93.9	33.5	184	21.9	60.4	30.1	32.1	84.5	75	30.4	25.7	45.2	17.7	60.9	26.3	48.6	46	71.1	92.4	34.1	53	74.2
10/13	45.1	60.7	77.8	16.6	91.6	40.9	32.5	15.2	93.3	32.9	180	21.8	60	29.9	31.9	83.9	73.9	30.2	25.7	44.8	17.4	60.6	26.3	47.8	45	70.1	91.1	33.8	52	72.6
10/12	45.8	61.4	78.1	16.4	91.6	41.1	32.7	15.3	96.2	33.2	180	21.5	60.1	31.4	31.9	83	74.5	30.1	25.5	44.6	17.5	60.9	26.2	48.8	45	70.7	89.7	33.5	53	73.3
10/11	44.6	61.1	77.1	16.2	91.5	40.6	31.6	15.1	93.8	33.2	179	21.4	59.7	30.5	31.6	83.9	72.6	29.5	25.5	43.6	17.5	60.6	26	47.8	45	70.3	89.1	33.3	52	72.5
10/10	44.3	61.1	75.6	16.3	92	40.3	32	15.1	93.3	33.1	180	21.3	60.2	30.5	31.6	83.1	72.9	29.8	25.4	44.1	17.6	60.8	26.3	48.4	45	70.3	87.8	33.7	52	72.5
10/7	42.3	59	72.2	15.9	88.5	39.1	30.7	14.5	89.9	32.4	176	20.7	59	29	31.2	81.9	70.2	29.2	24.8	42.8	17.1	59.9	25.7	46.4	44	68.1	84.9	33	51	69.9
10/6	43.3	58.7	73.7	16	88.7	39.1	31	14.5	95	31.9	176	20.5	58.7	30.6	30.9	81.8	70.9	29	24.9	43.4	16.9	59.7	25.6	47.6	44	67.6	85.8	32.8	50	70.2
10/5	42.3	57.3	70.9	15.4	88.7	38.5	30.5	14.3	91.5	31.4	171	20.3	58.2	29.1	31	80.6	68.9	29	24.4	42.7	16.7	59.4	25.5	47	43	66.3	83.7	32.6	50	70.3
10/4	42.1	56.5	69.3	14.8	85.7	37.6	28.9	13.9	91.8	31	169	19.7	58.1	28.6	30.8	81.2	68.4	23.9	23.9	41.6	16.3	59.4	25.2	45.6	42	66.3	83.4	32.8	50	69.2
10/3	42.2	55.6	67.4	14.4	84.2	36	28.1	13.7	87.4	30.2	167	19.2	58	26.8	30.9	80.8	67.4	29.1	23.2	40.3	16.1	58.9	25	44.4	42	66.1	82.9	32.7	49	67.6
9/30	43.5	57.8	70.5	14.7	86.8	37.3	29.2	14.2	91.8	31.4	169	19.8	59.5	28.2	31.9	82.5	68.2	30.2	23.5	41.5	16.4	59.3	25.3	46.2	45	67	84.4	33.1	49	69
9/29	45.3	59.6	72	15	88.5	38.6	29.7	14.8	96.9	32.3	173	20.7	59.7	29.4	32.6	83.4	70.8	30.1	24	42.7	16.7	59.7	25.6	47.1	45	69	86.3	33.4	49	70.2
9/28	45.1	59.1	71.7	15	86	38.2	29.5	14.4	93.5	32.1	172	20.7	59.1	28.5	32.2	82.7	69.9	29.5	24.2	43	16.3	58.8	25.4	45.7	46	68.5	86.3	33.2	49	68.5
9/27	45.7	59.9	74.2	15.2	87.7	39.5	30.2	14.7	96.6	32.4	172	21	59.6	29.6	32.9	84.3	72.5	29.7	24.2	44	16.5	59.3	25.5	46.7	48	69.7	88.1	33.2	50	69.3
9/26	46.1	59.2	73.4	15.2	85.7	38.8	29.4	14.6	96.2	32.5	169	20.7	58.5	29.6	32.5	83.9	71.5	29.2	24	43.4	16.5	58.7	25.2	45.8	47	68.2	83.9	32.7	49	68.1
9/23	45	56.8	70.6	14.8	84.3	37.8	28.9	14.2	92.4	32.2	164	20.6	57.5	27.7	31.9	82	70.3	28.7	23.7	43.1	16.2	57.4	24.7	44.7	46	65.7	88.7	32.3	48	65.9
9/22	45.2	56.1	70.6	14.5	84.5	38.9	28.6	14.1	91.2	31.6	163	20.1	57.8	27.4	32.1	80.7	69.3	28.7	23.7	40.9	16.3	57.4	24.7	44.9	46	65.1	86.9	32	48	65.8
9/21	46.8	58.3	75.8	15	88.3	41.7	30.3	14.4	95	32.4	167	20.4	59	28.4	32.7	82.2	72.6	29.4	24.5	41.7	16.6	59.1	25.1	45.7	47	71.3	90	32.3	49	68.4
9/20	47.8	60.7	79.9	15.7	91.6	42.9	31.3	15	99.6	33.3	169	20.6	60	30.2	33.4	83.9	75.3	30	25.5	43.5	17	60.1	25.6	47.8	48	72.4	91.4	32.8	50	70.3
9/19	47.2	61.3	80.8	15.7	92.5	43.3	31.5	15.1	102	32.8	167	20.4	59.9	30.4	33.3	83.4	75.3	29.8	25.7	43.8	16.8	59.8	25.4	47.3	48	72	88.6	32.7	50	70
9/16	48.6	62.4	82.1	15.8	93.4	44	31.9	15.3	104	33.1	167	20.4	60.3	31.3	33.7	82.9	76.5	30.2	25.6	43.6	16.9	60.3	25.7	48	49	71.9	89.4	33.1	50	70.8
9/15	47.9	61.4	82.7	15.8	93	43.4	31.9	15	105	32.8	164	20	60.1	31.7	33.6	82.7	76.6	30	25.5	42.8	17.2	58.9	25.4	47.2	49	72	89.1	32.6	50	70.3
9/14	47.5	60.2	81.6	15.5	91.2	42.5	31.3	14.6	101	32.1	162	19.6	59.5	30.7	33	81.5	75.5	29.6	25	42.5	17.1	58.5	25.1	46.4	47	70.2	88.2	32	50	69
9/13	46.5	60	81.2	15.5	89.9	41.7	30.5	14.3	101	31.2	158	19.3	59.4	30.4	32.7	80.9	73.9	29.3	24.6	41.7	17	58.1	25	46.4	47	69	86.4	32	49	68.1
9/12	46	59.6	80.1	15.3	89.9	41.4	30.3	13.9	99.9	30.9	157	18.9	59.4	30.4	32.6	80.9	74.3	29.2	24.4	41	17	58	24.8	46.4	45	67.8	86.1	31.7	49	68.3
9/9	45.9	59	80.2	15	89.2	42.3	30.1	14	99.2	30.5	156	18.3	59.4	30	32.6	79.8	72.8	29	24.3	40.1	17	58	24.5	45.6	44	67.2	85	31.7	49	67.5
9/8	48	60	83.2	15.5	92.2	43.3	31	14.4	102	31	160	18.5	60.7	31.4	33.4	83.2	75.3	29.9	24.8	41.5	17.5	59	24.8	46.8	46	69.2	86.1	31.8	50	69.2
9/7	48.5	62	84.7	15.1	93	44.3	31.6	14.6	105	31.4	162	18.7	61.1	32.6	33.2	83.8	76.6	30.1	24.5	42	17.7	58.8	25	47.6	46	69.5	87.1	32.1	50	70
9/6	46.6	59.9	81.9	14.5	89.6	43.1	30.8	14.1	101	30.6	160	18.2	60.4	31.3	32.6	83.4	74.2	29.4	24.1	40.7	17.3	58.5	24.7	45.6	44	67.2	84.5	31.7	49	67.6
9/2	47	61.1	81.6	14.6	90.3	43.7	31.4	14.6	104	30.8	161	18.3	59.8	32.4	32.7	83.7	75.4	29.5	24.4	40.8	17.2	58.7	24.9	45.9	44	67.7	84.2	32	50	68.5
9/1	48	63.1	84.6	15	92.3	44.7	32.3	15	109	31.5	165	18.6	61	34	33.1	84.6	77.6	30	24.7	42	17.6	59.3	25.1	47.2	46	69.6	86.4	32.3	50	69.8

Figure C.4: The historical stock prices of stocks during 2011

Date	AXP	BA	CAT	CSCO	CVX	DD	DIS	GE	GS	HD	IBM	INTC	JNJ	JPM	KO	MCD	MMM	MRK	MSFT	NKE	PFE	PG	T	TRV	UNH	UTX	V	VZ	WMT	XOM
8/31	48.2	63.8	86.9	14.9	92.6	45.1	33	15.1	113	31.9	166	18.7	61.4	35.2	33.1	84.9	78.9	30.2	25.1	41.9	17.6	59.7	25.3	47.4	46	70.7	86.5	32.6	51	70.3
8/30	47.2	63.1	85.8	14.8	92.2	45	32.5	14.9	112	32.1	167	18.8	61.4	34.7	32.8	85.2	78	29.8	24.8	41.8	17.5	59.6	26.3	47.2	46	70.3	86.5	32.7	50	70.2
8/29	47.1	61.7	84.2	14.9	92.5	44.4	32.1	14.9	112	32.3	167	18.9	61.5	35.2	32.7	84.7	78	29.6	24.4	42.9	17.5	59.2	26	47.7	45	70.4	85.2	32.5	51	70.4
8/26	47	60	81.4	14.5	90.8	43.1	31.4	14.4	108	32.3	163	18.4	60	33.9	32.2	83.9	76	29.1	23.8	41.8	16.9	58.7	25.8	45.4	44	68.2	84.5	32.2	50	69
8/25	46.6	58.3	79.5	14.3	89.9	42.5	31	14.3	106	32.1	160	18.1	59.7	33.4	31.9	82.7	74.3	29	23.2	40.5	16.7	58.6	25.8	45.1	44	67.4	82.8	32.2	50	68.2
8/24	46.6	58.9	81.6	14.7	91.4	43.1	31.5	14.6	107	32.6	161	18.4	60.7	33.6	32.7	84.1	76.3	29.4	23.5	41.5	17.1	59.4	26.1	46.5	44	68	85	32.8	51	69.9
8/23	45	58	79.3	14.7	91.2	42.4	31.1	14.4	103	31.4	159	18.3	60.1	32.6	32.4	83.5	75.7	29.1	23.3	40.5	17	59.1	25.7	46.4	43	66.6	82.5	32.4	51	70
8/22	43.3	55.7	76.3	14.2	87.4	40.9	31	14	103	30.6	154	18	58.6	31.3	31.6	81.9	73.1	28.5	22.6	38.4	16.4	57.9	25.3	46.2	42	64.5	78.5	31.3	50	66.7
8/19	43.1	54.9	76.4	14.3	87.4	40.9	30.9	14	108	30.3	152	17.8	58.4	32.2	31.5	81.4	73.1	28.5	22.7	38	16.4	57.2	24.9	46.5	42	64.3	78.3	31.2	50	66.3
8/18	42.9	56.3	79.6	14.2	87.4	41.5	31.5	14.2	109	30.5	158	18.4	58.5	33	31.8	79.9	73.6	28.3	23.3	38.7	16.5	57.1	25.3	46.8	42	64.9	79	31.6	49	67.4
8/17	44.5	59.4	83.7	15	91.5	43.7	32.3	15	113	31.7	166	19.2	59.5	34.2	32.5	81.6	76.9	29.4	23.8	40	17.2	57.8	25.9	49.1	44	68.6	82.7	32.1	49	70.5
8/16	43.5	59.4	85.4	15.2	91.2	43.8	32.4	15	113	31.4	165	19.3	59.6	33.7	32	80.8	77.5	29.3	23.9	40.3	17	57.8	25.6	48.9	44	68.7	84.3	31.4	49	69.8
8/15	44.4	59.9	87.3	15.2	92.1	44.6	32.6	15.2	115	29.9	167	19.4	59.8	34.5	32	81	78.6	29.3	23.9	41.2	17	58	25.6	49.1	44	69.6	84.4	31.5	48	70.6
8/12	43.5	59	85.8	15.2	89.1	44.2	32.1	14.7	113	29	162	19.2	58.6	33.6	31.5	80.7	77.9	28.6	23.6	40.8	16.6	57.3	25.1	48.1	43	68.6	82.4	30.9	47	68.4
8/11	43.7	56.2	83.4	15.1	87.5	43.9	31.1	14.5	114	28.5	161	19.3	58.7	34.4	31.2	80.5	76.7	28.4	23.6	40.3	16.6	56.7	25.3	48.3	42	66	83	31.2	47	68
8/10	41.5	54.8	79.8	13	84.2	41.5	30.6	14	107	27.1	157	18.5	55.7	32.2	30	78.4	73.8	27.2	22.7	38.4	15.8	54.9	24.8	46.1	40	63.8	77.9	30.3	46	64.6
8/9	44.7	59.1	83.6	13.3	86.8	43.2	33.6	14.8	119	28.6	165	19.2	57.6	34.1	31.3	80.2	78.1	28.5	24	39.9	16.4	56.5	25.6	48.6	43	67.7	81.3	30.9	48	67.6
8/8	41.7	55.7	78.9	13.2	83.9	40.8	32	14.3	114	27.5	161	18.7	56.6	31.9	30.6	76.6	74.2	27.3	23	38.3	15.5	55.6	24.6	45.7	41	66	77.9	29.8	46	66.3
8/5	45.8	59.5	86.9	14.2	90.7	43.8	34.1	15.3	121	29.2	166	19.3	58	35.2	31.3	79.4	78.1	28.9	24.1	40.6	16.3	56.8	25.7	49.4	44	70.2	82	31.5	48	70.6
8/4	45.4	59.8	85.6	14.1	90	43.7	34.2	15.3	122	30.1	165	19.4	57.2	35.5	30.9	78.6	77.6	28.7	24.3	39.5	16.1	55.9	25.6	50.1	44	70.4	83.5	31.4	47	69.7
8/3	47.4	63.9	92	14.7	95.5	46.7	36.3	16.2	128	31.3	172	20.3	58.7	37.4	32.1	79.8	81.4	29.8	25.3	42.2	16.8	57	26.2	51	46	75.1	86	32.4	48	73.4
8/2	47.1	64.2	92.8	14.7	96.2	46.3	35.8	15.9	127	31.1	171	20	58.7	37.3	31.5	79.3	80.5	30	25.1	41.2	16.7	57.1	25.9	50.9	45	74.7	82.1	31.9	49	73.5
8/1	48.5	66.7	96.2	15	98	47.7	37.2	16.6	130	32.5	174	20.5	59.6	37.9	31.8	80.6	81.9	30.5	25.6	43.2	17.5	57.6	26.2	51.3	46	77.9	84.2	32.3	50	75.1
7/29	48.5	66.8	94.4	15.2	96.7	47.6	37.4	16.6	131	33.1	175	20.6	60	37.9	31.9	80.7	82.3	31.1	25.7	43.6	17.7	57.7	26	51.8	48	78.4	84	31.8	50	75.3
7/28	49	67	95.2	15.2	97.6	48.4	38.2	16.8	131	33.4	175	20.8	60.2	38.1	32.3	80.9	82.7	31.8	26	43.8	17.8	58.1	26	52.2	48	78.5	85.4	32.1	50	76.9
7/27	49	67	96.8	14.9	98.2	48.4	38.3	16.8	130	33.8	174	20.8	60.4	38.1	32.2	81.2	83.1	32.1	25.6	42.9	17.7	58.3	26.6	52.9	48	79.3	86.2	32.6	50	78.6
7/26	50	66.5	101	15.5	100	49.7	39.2	17.2	133	34.5	176	21.1	61	38.8	32.5	82.1	84.9	32.6	26.3	43.7	18	59.2	26.6	53.4	50	81.6	87.6	32.7	51	79.6
7/25	50.2	67.6	101	15.4	100	50.1	39.3	17.6	132	34.8	177	21.2	61.3	39	32.5	82.2	89.7	32.7	26.2	44	18.2	59.4	26.6	53.2	50	82.6	87.5	32.6	51	79.8
7/22	50.7	68.9	100	15.6	101	50.8	39.4	17.6	131	34.7	178	21.3	61.8	39.5	32.7	82.6	90	32.9	25.8	44.4	18.4	60.3	26.9	54.1	51	83	88	33.1	52	80.4
7/21	51	69.1	107	15.5	102	50.7	39.5	17.7	131	34.8	178	21	61.5	39.6	32.6	80.7	90.5	33	25.4	44.5	18.5	60.5	26.9	54.2	50	83.5	86.5	33.8	51	80.3
7/20	50.5	68.4	105	15	99.6	50.2	38.1	17.4	128	34.6	177	21.2	61.3	38.4	32.3	80.5	89	32.4	25.4	43.9	18.3	60.2	26.9	53.6	50	82.5	87.1	33.6	51	78.6
7/19	50.2	66.9	105	14.9	99.9	50.1	38.3	17.2	124	34.3	178	21.2	61.8	37.8	32.5	80.4	89	32.6	25.8	43.9	18.3	60.1	26.9	53.6	50	84.1	87.7	33.3	51	78.9
7/18	49.8	66	103	14.7	98.8	49.5	37.5	16.9	125	33.9	169	20.5	62.1	37.3	31.5	79.7	89.3	32.4	24.9	43.9	18.1	60.1	26.8	53.1	50	82.9	86.5	33	50	78
7/15	50.2	67.6	104	14.8	98.7	50.1	38	17.1	126	34.1	169	20.6	62.4	37.4	31.7	79.7	90.1	32.8	25.1	44.4	18.2	60.3	26.9	54.4	50	83.6	87.2	33.1	51	78.4
7/14	49.8	67.5	102	14.6	97.3	49.8	38.3	17.2	126	34.1	168	20.5	62.6	37.8	31.8	80	89.7	33.1	24.8	44.5	18.3	60.1	27.2	54.3	51	83.1	86.4	33.2	51	77.6
7/13	50.4	68.4	103	14.8	97.7	50.5	38.3	17.1	126	34.3	168	20.7	62.6	37.1	32	79.2	90.8	32.8	25	44.6	18.4	60.1	27.4	55	50	83.4	87.2	33.2	51	77.9
7/12	50.8	68.2	102	14.8	97	50.3	37.9	17	126	34.3	167	20.7	62	36.9	31.9	79.3	90.5	32.5	24.9	44.2	18.4	60	27.3	54.8	50	83.7	86.2	33.4	51	77.3
7/11	50.7	69.6	103	14.6	97.1	50.7	38	17.3	128	34.5	168	21	62.3	36.9	32	79.6	91.2	32.5	25	44.3	18.4	60.2	27.4	54.7	49	84.4	86.4	33.4	51	77.3
7/8	51.5	71.2	105	14.9	98.4	51.3	38.7	17.6	130	34.8	170	21.3	62.5	38.2	32.2	79.8	92.2	32.9	25.3	45.1	18.5	60.4	27.7	55.1	50	85.5	88.2	33.7	51	77.8
7/7	52	72.1	106	15.1	99.1	51.7	38.5	17.9	131	35.2	170	21.4	62.9	38.7	32.3	80.3	92.5	32.6	25.1	45.3	18.6	60.4	27.7	55.4	51	86.5	88.6	33.7	52	77.7
7/6	51.2	70.9	105	14.8	97.7	51.1	38.3	17.6	130	34.7	171	21	62.5	38	32.2	80.1	91.8	32.4	24.7	44.7	19.1	60.2	27.7	55.1	51	85.6	86.7	33.8	51	77
7/5	50.8	70.3	103	14.9	97.7	50.4	38.7	17.6	130	34.7	169	20.7	62.6	38.4	32.1	79.9	90.8	32.4	24.4	44.4	19.1	59.8	27.7	55	51	84.7	86.8	33.6	50	77

Figure C.5: The historical stock prices of stocks during 2011

Date	AXP	BA	CAT	CSCO	CVX	DD	DIS	GE	GS	HD	IBM	INTC	JNJ	JPM	KO	MCD	MMM	MRK	MSFT	NKE	PFE	PG	T	TRV	UNH	UTX	V	VZ	WMT	XOM
7/1	50.8	70.4	103	15	96.8	50.4	38.5	17.8	132	34.9	168	20.8	62.3	38.9	32	79.9	91.3	32.5	24.4	44.4	19.1	59.8	27.8	55.6	51	85.3	86.4	33.6	51	77.4
6/30	50.1	70.1	101	14.8	95.6	50	37.8	17.5	129	34.4	165	20.4	61.6	38.1	31.6	78.6	89.5	32.2	24.4	43.6	18.9	59.1	27.5	54.9	50	83.8	82.8	33.1	50	76.8
6/29	49.4	69	98.3	14.5	94.2	49.4	37.2	17.2	128	34.1	164	19.7	61.4	37.6	31.3	78.9	87.9	32	24	43.5	19	58.2	27.4	55	50	81.8	85.1	32.6	50	75.8
6/28	48.1	68.4	98.8	14.3	93.3	48.7	36.7	17.1	125	34.2	164	19.8	61	36.8	31	78.7	87.8	31.8	24.2	43.5	18.9	58.3	27.2	54.1	50	81.9	74	32.5	50	75.2
6/27	47.9	67.9	95.9	14.2	91.9	47.9	36.9	16.9	126	33.4	161	19.7	60.4	37.1	30.6	76.8	87.1	31.5	23.6	39.5	18.5	58.4	27	53.8	49	80.8	72.6	32.3	49	73.6
6/24	46.7	67.6	95.1	14.1	91	48.1	36.4	16.6	127	33.3	159	19.5	60.2	36.8	30.5	76.3	85.8	31.5	22.8	39.3	18.5	58.2	26.7	53.3	49	79.8	72.1	32	50	72.5
6/23	47.6	67.6	95.7	14.6	92.4	47.5	36.6	17	128	33.8	160	20	60.8	37.3	30.5	76.8	87.2	31.9	23.1	39.8	19	59	26.9	53.7	49	80.6	74	32	50	74
6/22	48.2	68.4	95.3	14.5	94	47.5	37.1	17.2	130	33.1	159	19.7	61.2	37.9	31.2	77.1	87.6	32.3	23.1	39.7	18.6	59.6	27.1	53.9	50	80.9	73.4	31.9	50	75.3
6/21	47.9	70.2	96.5	14.7	94.4	47.9	37.6	17.4	132	33.6	160	19.9	61.5	38.1	31.1	77.2	88.2	32.6	23.2	40.9	18.8	59.7	27.3	54.6	51	81	73.7	32	50	76.1
6/20	47.2	70.7	93.4	14.3	92.9	46.6	37	17.1	131	33	159	19.6	61.6	37.7	30.9	77.1	87.4	32.6	23	40.3	18.6	60.6	27.1	54.1	50	81	72.8	31.7	50	75.2
6/17	46.9	70.3	91.3	14.2	92.2	45.9	36.9	17.1	133	32.8	158	19.5	61.4	38	30.8	77	86.6	32.3	22.8	39.3	18.6	60.2	27	54.4	48	80	73.1	31.5	50	74.6
6/16	46.8	70.2	90.8	14.2	92.4	45.7	36.8	17.1	132	32.7	156	19.7	61.4	37.6	30.7	76.3	86.4	32.1	22.5	38.9	18.6	59.8	26.7	54.3	48	79.8	73.6	31.3	50	74.8
6/15	45.7	70	91	14	91.5	45.9	37.2	16.9	130	32.2	156	19.7	61.2	37.9	30.5	75.8	85.9	32.1	22.3	38.9	18.6	59.3	26.6	54.1	48	78.8	73.5	31.2	49	74.3
6/14	46.5	70.8	93.1	14.3	93.5	46.6	37.4	17.1	133	33	158	20.1	62.1	38.7	30.8	76	87.4	32.5	22.7	39.8	18.9	60.2	27	55.1	48	79.5	74.6	31.6	50	75.9
6/13	46.6	69.2	90.8	14.2	92	45.8	37.2	16.9	133	31.6	157	19.7	61.6	38.8	30.6	75.3	86.2	32.2	22.6	39	18.8	60.3	26.8	55.2	47	78.3	73.6	31.7	50	74.8
6/10	46.1	68.9	92.1	14.3	92.7	46.1	37.3	16.8	131	31.5	157	19.7	61.2	38.2	30.5	75	85.6	32	22.2	38.6	18.5	60.2	26.6	55.6	48	78.3	73.4	31.3	50	75.3
6/9	46.7	70.4	94.4	14.5	94.1	46.7	38.1	17.1	129	32.3	159	20	61.8	38.1	30.7	75.8	86.8	32.5	22.5	39.1	19.1	60.5	26.6	57.4	49	79.6	75.1	31.7	51	76.6
6/8	46.5	70	93.3	14.5	92.9	46	38	17	127	32.1	158	20.1	61.2	37.6	30.5	75.7	85.8	32.3	22.5	38.8	19	60.3	26.6	57	47	78.6	75.4	31.9	51	76.2
6/7	47.2	70.4	95	14.7	92.5	46.6	38.1	17	129	32.3	157	20.3	60.6	37.9	30.4	75.7	85.8	32.1	22.6	39.6	19.1	60.5	26.6	57.2	46	78.8	78.4	31.4	51	75.5
6/6	47.1	70.7	95	15.1	92.7	46.1	38.2	17	130	32.3	158	20.1	61.1	37.7	30.5	75.3	85.5	32.1	22.5	38.6	19.1	60.9	26.6	57	47	79.2	76.8	31.3	51	75.8
6/3	47.6	71	96.2	15.1	93.9	46.6	38.1	17.3	131	32.6	159	20	61.2	38.7	30.5	75.1	85.6	32.3	22.4	38.9	19.2	60.9	26.9	57.1	47	78.8	77.8	31.7	51	76.6
6/2	48.4	71.8	97.2	15.4	94	47.4	38.8	17.5	130	33.1	160	20.3	61.5	38.7	30.8	75.3	86.3	32.6	22.7	39.3	19.3	61.4	27.1	57.1	48	79.8	79.3	32.1	51	76.8
6/1	48.2	71.5	96.3	15.5	95.3	47.7	39.1	17.6	132	33.3	160	20.3	61.5	38.9	31.1	75.5	86.3	32.7	22.9	39.6	19.3	61.8	27.3	57.1	47	79.8	77.8	32.2	51	77.4
5/31	49.9	74	101	15.9	97.5	49.3	40.3	18	136	34.2	162	20.7	62.3	40.2	31.1	76.1	89.1	33.1	23.5	40.7	19.7	62.3	27.7	58	47	83.1	79.6	32.8	52	78.8
5/27	49.4	73	99.5	15.6	95.9	48.7	40.2	17.9	134	33.9	161	20.5	61.8	39.8	31	76.1	88.2	32.7	23.2	40.8	19.2	61.6	27.4	57.4	47	81.7	78.4	32.6	52	78
5/26	49	72.7	99.1	15.4	96.1	48.1	39.7	17.8	131	34.1	161	20.7	60.6	39.5	31.1	76.3	87.6	32.8	23.1	40.5	19.2	61.6	27.3	57.1	46	81.8	77.9	32.5	52	77.8
5/25	48.7	72.4	98.2	15.3	96	48.3	39.8	17.7	131	34.5	161	20.9	60.8	39.3	31.2	76.6	87.5	33.1	22.7	40.1	19.1	61.8	27.2	57.4	46	81.6	77.4	32.3	52	77.4
5/24	48.9	71.7	96.6	15.4	95.1	47.4	39.8	17.5	132	34.4	162	20.8	60.4	39.4	31.4	76.5	86.8	33.3	22.7	40.7	18.9	62.3	27.3	57	46	81	76.2	32.8	52	76.7
5/23	49.4	72.3	96.9	15.5	94.2	47.8	39.9	17.8	131	34.7	162	21.1	60.2	39.6	31.5	76.4	87.3	33.3	22.7	40.3	18.9	62.4	27.2	57.4	47	81.5	75.6	32.7	52	76.2
5/20	49.5	73.5	99.3	15.6	95.4	48.8	40.2	18	130	34.9	164	21.4	60.3	40.1	31.8	76.2	88.3	33.4	23	41	19	62.7	27.4	58.1	48	82.8	76.9	33	52	77
5/19	50.1	74	100	15.7	96.6	49.5	40.1	18.3	134	35.2	164	21.7	60.9	41	31.9	76.4	89.4	33.7	23.2	41.5	19.3	62.8	27.5	58.6	49	83.4	78.3	33.2	52	77.7
5/18	49.3	73.2	100	15.7	95.6	49.3	40.1	18.2	136	35.3	164	22	61	41	31.8	75.5	88.7	33.9	23.2	40.9	19.5	62.7	27.3	58.8	49	82.5	78.6	33	52	77.2
5/17	49	72.7	97.1	15.7	93.3	48.4	39.7	18	136	35.2	164	21.7	60.9	40.8	31.7	74.9	88.1	33.6	23	40.9	19.4	62.7	27.3	58.4	48	82.7	79	33	53	75.9
5/16	48.4	73.8	101	15.7	93.8	49	39.7	18.2	136	34.8	162	21.8	60.9	39.9	31.6	75.1	89.6	33.6	22.9	41.1	19.3	62.3	27.2	58.8	48	83.7	78.4	32.8	53	75.7
5/13	47.8	75	101	16	94.5	49	40.2	18.3	137	34.9	163	21.6	61.1	40.2	31.8	74.8	90.1	33.4	23.3	41.6	19.2	62.2	27.5	58.2	48	83.7	78.4	33.1	53	76.3
5/12	47.9	75.3	103	16	94.8	49.9	40.3	18.5	138	35.1	166	21.8	61.7	41	31.8	74.7	90.7	33.6	23.6	40.8	19.2	62.2	27.7	59.6	48	84.3	78.4	33.3	53	76.5
5/11	48.2	75	105	16.8	94.3	50.1	40.2	18.5	143	35.1	163	21.6	61.1	41.2	31.4	73.6	89.6	33	23.6	39.9	18.9	61.3	27.5	59	48	84.3	77.8	33.1	52	76.6
5/10	48.5	75.4	107	16.8	96.3	51.4	42.5	18.6	145	35.1	164	21.2	60.3	42	31.4	73.8	89.8	33.2	23.9	40.3	19	61.1	27.8	59.6	49	84.5	78.8	33.4	52	78.2
5/9	48.5	75.1	106	16.6	96	50.9	41.8	18.4	144	34.8	163	21	60.2	41.8	31.2	73.4	90.1	32.9	24.1	40	18.8	60.6	27.5	59.2	49	84.3	78.3	33.1	52	78.1
5/6	48.5	74.8	105	16.6	94.9	50.2	41.7	18.4	145	34.8	162	21.4	59.9	41.9	31.2	72.9	89.7	32.8	24.1	39.6	18.7	60.7	27.4	59.2	48	84	77.9	33.1	52	77.6
5/5	47.9	74	104	16.5	94.7	49.5	41.4	18.3	145	34.9	161	21.7	59.7	42	31.1	72.8	89.1	32.7	24	39.8	18.6	61.3	27.4	58.8	48	83.3	77.2	33	52	77.6
5/4	48	74.4	105	16.5	96.6	50.1	41.3	18.6	146	35	163	21.6	60.3	42.3	31.4	73.2	90.3	33	24.3	39.9	18.8	61.8	27.9	59.3	48	84.1	78.2	33.5	52	79.6

Figure C.6: The historical stock prices of stocks during 2011

Date	AXP	BA	CAT	CSCO	CVX	DD	DIS	GE	GS	HD	IBM	INTC	JNJ	JPM	KO	MCD	MMM	MRK	MSFT	NKE	PFE	PG	T	TRV	UNH	UTX	V	VZ	WMT	XOM
5/3	48.2	75	108	16.5	97.9	51	41.7	19	147	35.2	166	21.2	60.8	42.7	31.6	73.1	90.9	32.8	24.1	39.4	18.6	61.1	27.9	59.8	48	84.6	78.5	33.6	52	80.4
5/2	48	75	109	16.6	99.8	51.6	41.9	18.8	146	35.2	165	20.9	60.8	42	31.6	72.8	91	32.8	23.9	39.7	19.1	60.6	27.4	59.3	48	84.7	77.6	33.4	52	81.6
4/29	47.4	75.3	110	16.6	101	52.2	41.8	18.8	146	35	163	21.2	60.3	42.5	31.4	72.5	91.2	32.4	24.2	39.7	19.1	60.4	27.3	59.1	47	84.3	76.6	33.6	52	82.6
4/28	46.9	74.1	107	16.3	100	52.1	41.7	18.9	145	35.3	164	20.8	60	42.7	31.4	72.2	91.2	32.3	24.9	39.8	19	60	27.5	58.9	47	83.8	76.2	34	51	82
4/27	46	71.8	107	16.2	101	51.1	41.2	19	148	35.2	163	20.7	60.2	42.3	31.2	72.1	90.5	32.1	24.6	39.1	18.8	59.6	27.5	58.1	46	82.7	77.4	34	51	82.4
4/26	45.5	71.3	106	16.6	100	50.6	41	18.5	148	35	161	20.5	59.6	42	31.2	71.2	90	31.6	24.4	38.6	18.4	58.9	27.1	57.3	46	82.7	77.3	33.4	51	82.1
4/25	45.5	70.7	104	16.2	99.1	50.6	40.6	18.3	147	35.4	161	20	58.8	41.5	31.6	71.4	88.3	31	23.9	38.6	18.3	58.5	26.8	57.7	46	81.5	76.3	32.9	50	80.9
4/21	45.5	71.2	104	16	99.7	51.4	40.9	18.3	148	35.6	161	19.6	58.8	41.6	31.6	71.2	88.1	30.7	23.8	38.7	18	58.4	26.9	57.2	46	82	76.3	32.8	50	81.1
4/20	45.4	70.8	103	16	99.5	50.9	40.7	18.7	147	35.7	158	19.6	59.1	41.5	31.6	72.6	88	30.7	24	38.6	18.6	58.8	26.4	55.2	43	80.9	75.9	33.6	50	80.4
4/19	45.1	69	101	15.7	97.2	50.2	40.1	18.6	147	35.6	158	18.1	57.5	41.6	31.3	70.9	86.2	30.4	23.4	38	18.7	58.9	26.6	54.3	42	77.5	74.5	33.1	50	78.7
4/18	44.5	68.7	98.4	15.8	96.4	49.4	39.9	18.4	148	35.6	159	17.9	55.5	40.9	31.4	71.3	85.8	30.6	23.4	37.9	18.6	59	26.6	55.1	42	76.9	74.2	33.1	50	78
4/15	44.7	68.5	102	16.1	98	50.4	40.2	18.4	150	36	159	18	55.6	41.8	31.7	71.6	87.1	31.1	23.6	38	18.7	59.2	26.9	55.8	43	78.5	75	33.6	50	79.1
4/14	44.3	68.2	102	16.2	96.8	49.7	39.7	18.4	150	35.6	158	17.9	55.1	41.9	31.8	71.4	87.2	30.5	23.7	38.6	18.7	58.4	26.5	55.7	43	79	74.8	33.5	50	78.3
4/13	44.5	68.1	102	16.3	95.8	49.5	40.4	18.3	155	35.5	157	18.1	54.7	43	31.4	71.2	87.1	30.2	23.9	38.3	18.6	58.1	26.4	55.9	43	79.3	76	33.5	50	78.1
4/12	44.5	68.9	101	16.5	96.1	49.4	40.3	18.4	155	35.6	156	18.1	55	43.4	31.2	71	86.7	30.3	23.9	37.7	18.6	58	26.7	55.7	43	79	75.5	33.4	50	78.1
4/11	44.8	69.6	103	16.5	99.4	50.4	40.6	18.5	156	35.4	157	18.4	54.9	43.6	31.4	70.6	88	30.3	24.2	37.7	18.8	57.4	26.9	56.4	43	80	76.2	33.6	50	79.9
4/8	44.7	69.3	104	16.7	101	50.9	40.5	18.5	155	35.3	157	18.3	54.6	43.6	31.4	70.4	87.5	30.4	24.3	37.3	18.6	57.1	26.9	55.6	43	79.8	75.2	33.5	49	80.7
4/7	44.6	70.1	104	16.9	100	51.1	40.7	18.7	157	35.6	157	18.3	54.6	44.1	31.4	70.4	87.5	30.1	24.4	37.7	18.5	57.3	26.8	55.8	43	80.3	74.9	33.5	50	80.5
4/6	44.7	69.6	105	17.1	100	51.5	40.9	18.9	156	35.3	157	18.2	54.7	44.3	31.5	71	88	30.1	24.4	37.9	18.5	57	26.7	56	43	80.6	74.5	33.6	50	80
4/5	43.7	69.1	106	16.3	101	51.5	41.1	18.7	153	35.4	157	18	54.9	43.4	31.4	70.9	87.6	29.9	24	37.6	18.6	56.9	26.5	55.2	43	80.4	74.2	33.7	50	80.2
4/4	43.7	69.8	107	16.1	99.8	51.1	41.3	18.9	153	35.3	157	17.8	55.2	43.1	31.5	70.7	87.9	30	23.8	37.3	18.7	57.4	26.6	55.7	44	80.2	73.8	33.9	49	79.7
4/1	43.7	69.8	107	16.1	99.9	50.7	41.5	18.7	155	35.4	157	18	54.6	42.9	31.3	70.4	87.4	29.8	23.8	36.9	18.6	57.3	26.5	55.7	44	80.3	72.8	33.7	49	79.5
3/31	43.5	69.7	105	16.2	99.2	50.5	41.7	18.4	153	34.9	156	18.4	54.4	42.7	30.9	70.4	87.7	29.8	23.7	36.5	18.5	56.8	26.4	55.5	44	79.7	72.2	33.8	49	79
3/30	44.2	69.6	106	16.4	99.6	50.9	42	18.5	154	35.4	157	18.7	54.5	43	30.8	70.2	86.8	30	23.9	37	18.5	57.2	26.5	55.4	44	79.3	72.8	33.7	49	79.3
3/29	43.9	69.5	105	16.5	99	50.2	41.6	18.2	153	35.5	156	18.5	54.3	42.6	30.6	69.8	87.1	29.6	23.8	36.8	18.6	56.5	26	55.3	43	79.2	70.8	33.6	49	78.1
3/28	44	69.2	104	16.2	97.8	49.6	41.2	18.1	151	34.5	155	18.6	54.4	42.5	30.3	69.4	86.4	29.3	23.7	36.6	18.5	56	25.4	55	42	78.6	71.3	33.1	49	78.3
3/25	43.9	69.2	103	16.3	98.5	49.7	41.6	18.1	152	35.2	155	18.6	54.1	42.5	30.4	69.7	86.6	29.4	23.9	37	18.5	56.2	24.9	55.1	42	78.6	70.7	32.7	49	78.5
3/24	43.9	68.6	103	16.4	97.2	49.3	41.5	18.2	154	35.2	153	18.6	54.2	42.3	30.2	69.4	86.9	29.5	24.1	37	18.5	56.4	24.7	54.9	42	77.9	71.3	32.6	49	77.7
3/23	43.3	68.6	101	16.6	97.3	49.1	40.9	17.9	154	34.5	153	18.5	53.9	42.2	30	69	86.5	29.4	23.8	37.2	18.1	56.2	24.3	54.5	41	77	70.6	32.4	48	77.5
3/22	43.1	67.8	101	16.4	97.1	49.3	40.1	17.9	155	34.2	151	18.4	53.9	42.1	29.6	68.4	85.1	29.3	23.6	36.4	18.2	56.2	24.3	54.4	41	76.1	70.2	32.4	49	77.5
3/21	42.7	67.2	102	16.4	97	49.5	40.5	18.1	155	34.3	151	18.5	54	42.2	29.6	68.3	85.4	29.2	23.6	37.1	18.2	56.6	24.4	55	42	76.7	70.6	32	49	77.3
3/18	42.5	65.2	99.5	16.1	94.8	48.7	39.9	17.7	154	33.9	149	18.2	53.7	42.3	29.2	67.6	83.5	28.8	23.1	37.4	18.4	55.9	24.1	54.2	41	75.4	70	31.4	48	75.9
3/17	41.8	64.4	97.7	16	94.3	48.1	39.5	17.7	150	33.7	148	18.2	53.3	41.2	29	68	83.2	28.4	23.1	41.2	18.1	55.8	24	54.8	41	74.8	70.1	31	48	76.2
3/16	40.8	63.9	95.1	16.1	91.8	47.3	39.3	17.4	149	33.6	147	18.1	52.9	40.6	28.7	67.9	82.2	28	23.1	40.9	17.6	55.1	23.6	54.2	40	73.7	69.6	30.1	48	74.4
3/15	42	65.7	95.5	16.4	93.4	48.1	40.3	18	152	34.2	152	18.4	53.7	41.3	29.4	69.5	84	28.7	23.7	41.2	18	56	24	54.6	41	74.8	69.8	30.6	49	76.4
3/14	42.3	66.7	96.7	16.8	93	48.7	40.9	18.3	153	34.5	155	19	54.3	41.9	29.8	70.1	85.4	29.2	23.9	41.6	18	56.6	24.4	55	42	76	70.5	30.9	49	77.3
3/11	42.6	67.6	94.8	16.9	92.2	48.6	41.6	18.7	155	35	156	19.1	54.8	42.3	30.2	71	86	29.5	23.9	42	17.7	56.7	24.6	55	42	76.5	71.1	31.4	49	77.1
3/10	42.4	67.3	93.2	16.9	91.4	48.3	41.1	18.5	155	34.9	155	19	54.7	42.1	30.2	71	84.5	29.3	23.7	42.5	17.6	56.7	24.7	54.6	42	76.2	70	31.9	49	76.4
3/9	43.4	68	97	17.1	94.2	49.4	41.8	19	157	35.5	159	19.4	55.4	43.1	30.4	70.2	87.4	29.7	24.1	42.9	17.9	57.2	24.9	55.5	43	78	72.3	32.1	49	79.2
3/8	43.5	68	98.7	17.2	95.7	49.9	41.8	19	156	34.9	155	19.3	55.7	43	30.4	69.9	88	29.4	24.2	43	17.9	57.3	24.6	55.1	42	78.4	72.7	32	49	79.4
3/7	42.1	66.9	96.8	17.1	95	48.9	41.7	18.7	154	34.5	153	19.4	55.4	41.8	30.2	70.6	86.7	29.3	24	42.9	17.9	56.9	24.1	54.7	42	77.4	72.7	31.6	49	79.5
3/4	42.1	67.7	97.6	17.3	95.7	49.5	42.2	18.7	155	34.8	155	19.7	56	42.1	30.2	70.4	86.5	29.5	24.2	43.3	17.9	57.2	24.1	54.9	43	78	73.3	31.6	49	79.9

Figure C.7: The historical stock prices of stocks during 2011

Date	AXP	BA	CAT	CSCO	CVX	DD	DIS	GE	GS	HD	IBM	INTC	JNJ	JPM	KO	MCD	MMM	MRK	MSFT	NKE	PFE	PG	T	TRV	UNH	UTX	V	VZ	WMT	XOM
3/3	42.6	67.7	98.8	17.5	96.1	50.1	42.7	19.1	159	35.1	157	19.9	56	42.7	30.3	70.6	87.1	29.5	24.4	43.3	18	57.7	24.3	54.8	43	78.9	74.4	31.9	49	80.6
3/2	41.4	65.6	95.7	17.4	95.5	48.8	41.9	18.7	156	34.3	153	19.6	55.8	41.9	29.8	69.2	85.7	29	24.3	42.4	17.5	57.6	24.3	54.6	42	77.2	72.5	31.9	48	79.9
3/1	41.5	66.2	94.6	17.5	94.9	48.9	41.6	18.6	156	34.4	153	19.5	55.7	42.2	30	69.3	84.9	28.9	24.4	42.1	17.4	57.9	24.3	54.9	41	77.3	71.3	31.6	49	79.6
2/28	41.9	67.9	97.5	17.5	95.7	50.4	42.4	19.2	158	35.1	155	19.6	56.4	43.2	29.6	70.1	86.6	29	24.8	42.8	17.5	58.2	24.5	55.6	41	78.6	71.6	32.4	48	80.3
2/25	41.9	68.2	96.6	17.6	94.2	49.7	41.6	19.1	159	34.7	155	20	54.7	43.2	29.8	68.9	84.7	28.7	24.7	42.3	17.2	58	24.3	55.3	41	78.5	73.2	31.6	48	80.1
2/24	41.9	66.8	95.3	17.3	94.1	48.7	41.1	18.9	157	34.7	154	19.5	54.8	42.5	29.6	69.1	84.5	28.6	25	41.6	17.2	58.2	24.1	55.1	41	77.9	72.5	31.2	49	80.7
2/23	41.8	66.3	94.8	17.3	94.3	49.4	40.8	18.6	157	34.9	153	19.3	54.9	42.5	29.6	69	84.7	28.6	24.8	41.4	17.1	59.2	24.2	56.1	41	77.9	71.5	31.4	49	81.7
2/22	42.6	66.9	96.6	17.5	92.5	50	41.3	19	157	35.6	155	19.9	55.2	42.6	29.5	69.5	86.3	28.8	24.8	41.9	17.2	59.1	24.4	56.2	41	78.6	72.3	31.6	50	80.2
2/18	43.8	68.9	100	17.8	91.1	51.4	42.2	19.6	162	36	158	20.2	55.6	44.4	29.9	69.9	87.2	29.3	25.2	42.7	17.5	59.3	24.7	56.5	41	80	74.4	32.1	52	79.3
2/17	44.1	68.2	97.9	17.6	89.6	51.1	42.3	19.6	161	35.7	157	20.1	55.3	44.3	29.9	69.8	87	29.5	25.4	41.3	17.6	59	24.6	55.5	41	79.6	74.7	31.9	51	78.7
2/16	45.1	68.4	98.1	17.5	89.2	50.2	42.3	19.6	163	35.4	156	19.9	55	44.4	29.3	69.8	86.6	29.4	25.2	41.2	17.6	58.8	24.5	55.2	41	80.1	74.5	31.7	51	78.6
2/15	44.5	67.4	97.6	17.6	88.9	49.7	41.7	19.6	162	35.3	156	19.6	55.1	43.3	29.2	69.9	85.8	29.2	25.1	41.1	17.3	59	24.4	55.1	40	79.5	74.2	32	51	77.9
2/14	44.8	68.2	97.8	17.7	89.4	50.1	41.9	19.6	161	35.2	156	19.7	55.2	43.1	29.2	70	85.7	29.3	25.2	41.2	17.3	59.5	24.6	54.6	41	79.8	74.3	31.5	51	79.7
2/11	45	68.1	98.1	17.6	88.3	50.2	42.1	19.5	160	35.1	157	19.9	55.2	43.1	29.4	69.9	85.6	29.5	25.3	41.4	17.1	59.7	24.6	54.7	41	79.8	73.5	31.9	52	77.7
2/10	44.8	68.6	95.3	17.8	88.5	49.5	42	19.4	159	34.9	157	19.9	55.4	42.1	29.4	69.6	84.4	29.5	25.5	41.7	17.3	59.6	24.4	54.2	41	79.4	73.3	31.9	52	78.1
2/9	44	68.5	94.5	20.8	88.1	49.1	42	19.4	160	34.8	158	19.6	55.4	41.8	29.2	69.7	84.7	29.5	25.9	41.9	17.3	59.4	24.2	54.6	40	78.6	71.9	32.2	53	77.5
2/8	44.2	68.2	95.3	20.7	89.5	49	39.9	19.4	162	34.8	159	19.8	55.4	42.3	29.1	69.2	83.5	29.7	26.2	41.7	17.4	59.6	24.1	53.7	40	78.6	73	31.9	53	77.9
2/7	43.1	67.5	95.2	20.7	89.4	48.6	39.7	19	161	34.2	157	19.8	55.4	42.1	28.9	67.5	82.8	29.4	26.1	41.6	17.3	59.6	24.2	53.1	40	78.3	72.4	31.6	52	78.4
2/4	42.2	67	94.4	20.8	88.9	47.9	39.4	18.8	159	34.4	156	19.8	55.3	41.3	29	68	82.4	29.3	25.7	41.2	17.6	58.7	24.2	53.3	41	77.3	71.4	31.9	52	77.3
2/3	41.9	66.6	93.7	20.6	89.1	47.4	39.2	18.9	159	34.3	156	19.7	55.3	42.1	29	67.8	82	29.3	25.6	40.3	17.5	58	24.2	53.1	41	77	70.1	31.9	52	77.9
2/2	42.1	66.6	93.9	20.4	88.3	47.3	39.2	18.9	159	34.2	156	19.5	55.1	42.1	29.1	67.6	81.9	30.2	25.9	40.3	17.3	57.9	23.9	52.6	40	76.9	70.6	31.7	52	77.9
2/1	42	65.9	92.9	20.2	88.1	47.1	38.6	19	159	34.6	156	19.5	55.1	42.5	29.2	67.5	81.9	30.3	25.9	40.2	17.3	58.1	24.1	52.7	40	76.6	69.2	31.8	53	78.3
1/31	41.8	65.2	91.9	19.9	86.9	46.2	37.7	18.4	158	34.4	155	19.4	54.4	41.6	29.1	67.7	82	29.6	25.7	39.6	16.4	58.2	23.8	52.2	39	76.1	68.4	31.2	52	75.3
1/28	42.2	64.9	90.7	19.7	85.5	45.9	37.6	18.4	156	34.3	152	19.4	54.6	41.2	28.8	67.3	81.6	29.5	25.7	39.1	16.4	59.2	23.8	51.8	39	76.3	68	31.3	53	73.8
1/27	42.9	66.2	91.6	20.2	86.8	45.8	38.2	18.5	158	35.5	154	19.7	55.2	41.7	29	68.3	83.2	29.6	26.8	39.8	16.6	59.2	24.3	52.5	41	77.3	69.1	32	54	74.6
1/26	42.8	65.7	90.7	20.2	86.7	45.9	38.2	18.2	155	35	154	19.7	55.1	41.6	29.1	69	83.6	29.6	26.7	40.4	16.5	61	24.8	52.1	39	76.2	69.5	31.9	53	74.4
1/25	43.1	67.8	89.4	20.3	86.1	44.7	38.6	18.2	156	34.8	154	19.5	55.5	41.5	29.1	69.3	82.6	29.7	26.4	39.8	16.6	61.5	24.9	52.2	39	76.5	70.1	31.4	53	73.5
1/24	44.1	68.2	89.6	19.9	86.2	44.6	38.7	18.3	160	34.3	152	19.2	56.6	41.7	29.3	69.2	84.3	30.1	26.3	39.7	16.6	61.4	24.6	51.6	38	76.3	70.3	30.9	52	73.4
1/21	44.3	67.2	87.9	19.5	85.9	44.1	38.5	18	160	34.2	148	18.9	57	41.9	29	68.9	83.3	30.2	26	39.5	16.5	60.8	24.5	51	39	75.1	68.7	30.7	52	73.7
1/20	43.6	66.7	88.7	19.6	84.9	43.9	37.9	16.8	160	34.1	149	19	57.2	41.4	29.1	69	82.1	30.4	26.3	39.9	16.4	60.7	24.4	50.9	39	74.5	69.2	30.4	52	72.6
1/19	43.5	67.3	90.5	19.6	85.1	44.6	37.9	16.7	160	33.3	148	19	56.9	40.5	29.3	69.2	82.1	30.2	26.4	40.2	16.5	60.3	24.5	50.5	39	74.8	67.7	30.4	51	73.1
1/18	44.6	68	91.2	20	85.5	45.3	38.2	17	168	33.7	144	19.1	56.5	41.4	29.4	68.6	82.2	30.2	26.6	40.4	16.6	60	24.5	50.7	39	74.5	69.7	30.1	51	73.5
1/14	44.5	65.7	88.7	20	85	45.4	38.1	17.2	169	33.6	143	19.1	56.9	41.6	29.2	68	82.2	30.5	26.2	40.4	16.5	60	24.6	50.7	39	74.1	69.6	31.1	51	72.7
1/13	43.4	65.5	88.8	19.9	84.4	45	38	17	165	33	142	19.3	57.2	41.1	29.3	66.7	82.1	30.9	26.1	39.7	16.4	60	24.3	50.7	38	74.5	69.6	31.4	51	71.6
1/12	43.3	65.8	88.2	19.9	84.6	44.8	37.9	17	165	32.6	142	19.3	56.8	41.4	29.2	67.6	82.7	33.1	26.5	40.2	16.5	59.5	24.2	50.7	37	74.4	71.1	31.1	51	71.5
1/11	43.4	64.7	88.6	19.7	84.1	44.7	38.2	17	163	32.6	140	19.1	56.6	40.4	29	67.9	81.8	32.9	26	40.4	16.4	58.7	24.1	50	38	73.9	70.4	31	51	70.7
1/10	42.8	64.8	88.1	19.6	82.8	44.7	38.3	16.9	163	32.2	141	18.7	56.5	40.2	29.2	67.6	81.3	33.2	26.2	40.4	16.5	58.9	24.5	50	37	73.5	70.3	31.5	50	70.2
1/7	42.7	65.1	88.4	19.8	83.5	45.4	38.2	16.8	164	32.2	141	18.7	56.9	40.4	29.1	68.3	80.4	33.3	26.5	40.1	16.5	59.1	24.9	49.5	37	74.1	71.4	31.5	50	70.6
1/6	43.1	64.5	88.2	19.7	83	45.6	38.4	16.9	166	32.2	142	18.8	57.5	41.2	29.2	68.1	80.4	33	26.7	40.3	16.4	59.2	25.2	50.5	37	74.1	71.6	31.8	50	70.2
1/5	43.4	63.3	89.1	19.6	83.7	45.8	38.7	17	168	32.3	140	19	57.6	41.4	29.4	68.6	80.8	32.6	25.9	40.6	16.3	59.3	25.5	51.3	36	74.2	70.6	32.6	51	69.8
1/4	42.1	62.8	88.4	19.3	83.9	45.5	37.8	17	167	32.4	141	19.2	57.6	40.9	29.6	68.2	80.8	32.4	26	40.4	16.2	59.5	25.4	51.6	36	74.1	69.1	32.2	51	69.9
1/3	41.6	62.3	88.8	19.3	84.2	45.6	36.6	16.7	167	33	141	18.9	57.1	40.3	30.2	70.3	81	32.1	25.9	41.4	15.9	59.3	25.3	51.8	36	74	69	31.5	51	69.6

Figure C.8: The historical stock prices of stocks during 2011