
Notes on the δ -expansion approach to the 2D Ising susceptibility scaling

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We study the scaling of the magnetic susceptibility in the square Ising model based upon the δ -expansion in the high temperature phase. The susceptibility χ is expressed in terms of the mass M and expanded in powers of $1/M$. The dilation around $M = 0$ by the δ expansion and the parametric extension of the ratio of derivatives of χ , $\chi^{(\ell+1)}/\chi^{(\ell)}$ is used as a test function for the estimation of the critical exponent γ with no bias from information of the critical temperature. Estimation is done with the help of the principle of minimum sensitivity and detailed analysis revealed that $\ell = 0, 1$ cases provide us accurate estimation results. Critical exponent of the sub-leading scaling term is also estimated.

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I. INTRODUCTION

Recently, the δ -expansion approach to the critical phenomena was studied within the high temperature expansion on the cubic Ising model [1]. The work has concentrated on the behavior of the inverse temperature β as a function of the mass parameter M , which is defined by the magnetic susceptibility χ and the second moment μ . The approach yielded results good enough for encouraging further exploration. The present paper extends the approach to the magnetic susceptibility in the 2-dimensional (2D) square Ising model. Within δ -expansion approach applied to the high temperature series, we try to recover the critical behavior of the susceptibility. Specifically, we will discuss unconventional way of estimating the exponent γ . The new point resides in the examination of the ratio $\chi^{(\ell+1)}/\chi^{(\ell)}$ ($\ell = 0, 1, 2, 3, \dots$) where $\chi^{(\ell)}$ stands for the ℓ th order derivatives with respect to $\log M^{-1}$,

$$\chi^{(\ell)} = \left(x \frac{\partial}{\partial x}\right)^\ell \chi, \quad x = 1/M. \quad (1)$$

The best choice of ℓ , the order of the derivatives, will be determined in the high temperature expansion. As will be explained in later (see (18) and (19)), the ratio

$$\mathcal{R}_\ell = \frac{\chi^{(\ell+1)}}{\chi^{(\ell)}} \quad (2)$$

converges to $\gamma/(2\nu)$ in the critical limit $x \rightarrow \infty$ for ℓ of which value is not restricted at least formally. In extracting the limiting value, we pay attention also to the best choice of ℓ for the estimation of $\gamma/(2\nu)$.

We like to remark the reader that, though χ is not obtained in a closed form in β , its mathematical structure has been studied and explored from long time ago in many prominent works (see the papers cited in [2, 3]). The value of critical exponents are exactly known, for instance, as $\nu = 1$ and $\gamma = 7/4$. The present work does not intend adding some new results along with the traditional direction. Rather, our motivation is in the series expansion *in the mass* (but not in the temperature) and to develop a new approach based upon the δ -expansion to the critical phenomena. Unlike ordinary expansion in the temperature variable, the representation of thermodynamic quantities in the mass has an advantage that no knowledge on β_c is required in the analysis of interested quantities. We even consider the temperature as function of the mass. Conventional approach takes thermodynamic quantities to be functions of $1 - \beta/\beta_c$. Then, to estimate critical quantities, one needs to know in the first place the precise value of β_c because the accuracy of β_c affects all estimation tasks. In the present approach, no such bias is present. As a typical example of application, we here employ the computation of the critical exponent γ . We extract the exponent $\gamma/(2\nu)$ from the critical behavior of χ expressed in $1/M$ series. We also stress that in the large mass expansion, the δ -expansion method plays a crucial role. By such a specific study, we expect a further development of the δ -expansion method in the application of the critical phenomena and, hopefully, in other branches of physics.

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It is also well-known in the square Ising model that $\beta_c = \frac{1}{2} \log(1 + \sqrt{2})$. The estimation serves one of the tests of our approach. The subject was studied in the same approach presented in [1] but the result is omitted in this paper. We just point out that using the mass as the basic parameter makes the estimation of β_c be just one of similar tasks, the estimation of the critical exponents and the amplitudes. The analysis presented in this work is a numerical one heavily based upon the high temperature series up to large orders, and widely applicable in other models beyond almost solved square Ising model.

The present work is organized as follows: In the next section, we briefly explain the series expansion approach to the susceptibility. The δ -expansion is introduced in this section and explain how it affects the behaviors of large mass expansion. We shall show that by the δ -expansion the scaling behavior is approximately observed in the large M expansion. In the next section, we estimate the critical exponent γ via the ratio of derivatives $\mathcal{R}_\ell = \chi^{(\ell+1)}/\chi^{(\ell)}$ ($\ell = 0, 1, 2, 3, \dots$). The estimation is carried out based upon the δ -expanded large mass series. Then, we conclude this paper.

II. SERIES EXPANSION AND δ EXPANSION

A. Series expansion at large and small M

Conventional definition of the susceptibility is given by

$$\chi := \sum_{all\ n} \langle s_0 s_n \rangle. \quad (3)$$

The behavior of χ near the critical point was investigated in many literatures and has long history to the present day (see [2]). A recent comprehensive study on the high temperature expansion was done in [4, 5] and we refer the works for review and main source of known results.

To begin with our discussion, let us briefly review the critical behavior of the susceptibility χ . Expansion around the critical point is conveniently done with the variable

$$\rho = (1/\sinh 2\beta - \sinh 2\beta)/2. \quad (4)$$

At the critical temperature $\beta = \beta_c = \frac{1}{2} \log(\sqrt{2} + 1)$, $\rho = 0$, and at high temperature where $\beta < \beta_c$, $\rho > 0$. At high temperature β near β_c , it has been settled that

$$\chi \sim const \times \rho^{-7/4} + const + const \times \rho^{1/4} + O(\rho \log \rho). \quad (5)$$

Since the variable ρ can be expanded in more conventional one,

$$\tau = \beta_c - \beta, \quad (6)$$

such that

$$\rho = 2\sqrt{2}\tau + 2\tau^2 + \frac{16\sqrt{2}}{3}\tau^3 + O(\tau^4), \quad (7)$$

we find

$$\chi \sim const \times \tau^{-7/4} + const \times \tau^{-3/4} + const + const \times \tau^{1/4} + O(\tau \log \tau). \quad (8)$$

Our approach uses the expansion of χ in terms of $1/M$. So it is necessary to rewrite the above behavior in terms of M . For the purpose we need series expansion of β in M at small enough M and the steps are explained below: The mass to be used in the investigation is not a priori known. There may be some candidates, the so-called exponential mass, second moment mass and maybe others. We have used in the previous work [1] the second one, since it is straightforwardly calculable in wide class of spin systems and field theories. However, in the 2D Ising model on the square lattice, the computation has been carried out just up to 25th order. The order is not so high for obtaining conclusive results. We therefore use the exponential mass, which is exactly obtained in [6],

$$\xi^{-1} = -\log \tanh \beta - 2\beta. \quad (9)$$

Some comments would be in order. We remind the reader that the above result is given in the large separation limit of the two point function where the spins are sited on a horizontal or vertical line. Here we do not test the

another candidate of the exponential mass which is defined by the diagonal correlation function where the spins are sited on a diagonal line. Further, we note that the counter part of ξ , the mass in momentum space, is given by $2(\cosh \xi^{-1} - 1)$. This quantity is in very close to the second moment mass defined by $4\chi/\mu$ which is more accessible to obtain in general models. For example the first 8 terms in the large mass expansions of β agree with each other. Relying upon the quantitative agreement of the two variable, instead of $4\chi/\mu$, we use basic parameter M defined by

$$M := 2(\cosh \xi^{-1} - 1). \quad (10)$$

Then, by using (9) and (10), we have

$$M = e^{-2\beta} \coth \beta + e^{2\beta} \tanh \beta - 2. \quad (11)$$

Use of (11) allows us expansion of β in $1/M$ to an arbitral large order. Around the critical temperature, we find

$$M = 16\tau^2 + 16\sqrt{2}\tau^3 + O(\tau^4). \quad (12)$$

Thus, inversion gives

$$\tau \sim \frac{1}{4}M^{1/2} + O(M), \quad (13)$$

and

$$\chi \sim \text{const} \times M^{-7/8} + \text{const} \times M^{-3/8} + \text{const} + \text{const} \times M^{1/8} + O(M^{1/2} \log M). \quad (14)$$

This is the goal which we aim recovering from large M expansion of χ .

Though the logarithmic term exists in (14) at order $M^{1/2}$, we neglect the presence. In our trial to the susceptibility scaling, we just assume the power like behavior near the critical point (the constant term in (14) is interpreted as the term with zero power),

$$\chi = C_1 x^{q_1} + C_2 x^{q_2} + \dots, \quad x := \frac{1}{M} \quad (15)$$

where $q_1 > 0$ and $q_1 > q_2 > \dots$. In ordinary term

$$q_1 = \frac{\gamma}{2\nu} = \frac{7}{8}, \quad (16)$$

and $q_2 = 3/8$ and $q_3 = 0$.

The derivative to the order ℓ is given by

$$\chi^{(\ell)} = C_1(q_1)^\ell x^{q_1} + C_2(q_2)^\ell x^{q_2} + \dots \quad (17)$$

and the ratio of derivatives behaves in the scaling region as

$$\mathcal{R}_\ell = q_1 + C_2/C_1(q_2/q_1)^\ell (q_2 - q_1)x^{q_2 - q_1} + \dots \quad (18)$$

Thus, at least formally, we find for any ℓ ,

$$\lim_{x \rightarrow \infty} \mathcal{R}_\ell = q_1. \quad (19)$$

It is interesting to note that, since $q_2/q_1 < 1$, larger ℓ makes the correction smaller. This leads us to expect that the convergence is faster for larger ℓ . Of course, large ℓ enhances higher order contribution which involves $(q_i/q_1)^\ell$ (i stands for some large integer). So, actually there is a limitation of such suppression mechanism at some ℓ . We like to study on the point under the approach within large M expansion and find a suitable value of ℓ for the estimation of the critical exponent q_1 .

We now turn to the series expansion at large M ,

$$\chi = 1 + \sum_{n=1} a_n x^n, \quad x = \frac{1}{M}. \quad (20)$$

The expansion of χ in β has been carried out to extremely large orders up to few thousands [4]. We here use first 100 terms. The substitution of β , which is given in series expansion in $1/M$ via (11), gives (20). The Table IV in Appendix shows the coefficients of the first 40 terms.

Then, for $\ell = 1, 2, 3, \dots$,

$$\chi^{(\ell)} = \sum_{n=1} n^\ell a_n x^n, \quad x = \frac{1}{M}. \quad (21)$$

The ratio of derivatives has expansion:

$$\mathcal{R}_0 = a_1 x + (2a_2 - a_1^2)x^2 + O(x^3), \quad (22)$$

$$\mathcal{R}_\ell = 1 + 2^\ell \frac{a_2}{a_1} x + \frac{2 \times 3^\ell a_1 a_3 - 2^{2\ell} a_2^2}{a_1^2} x^2 + O(x^3), \quad (\ell = 1, 2, 3, \dots). \quad (23)$$

It is interesting to see the convergence radius, approximately predicted by the ratio analysis of coefficients. As clearly shown in Fig1, \mathcal{R}_ℓ exhibits behaviors quite different for $\ell = 0, 1, 2$ and $3, 4$. The former cases show steady convergence to -8 but the later cases show no sign of convergence.

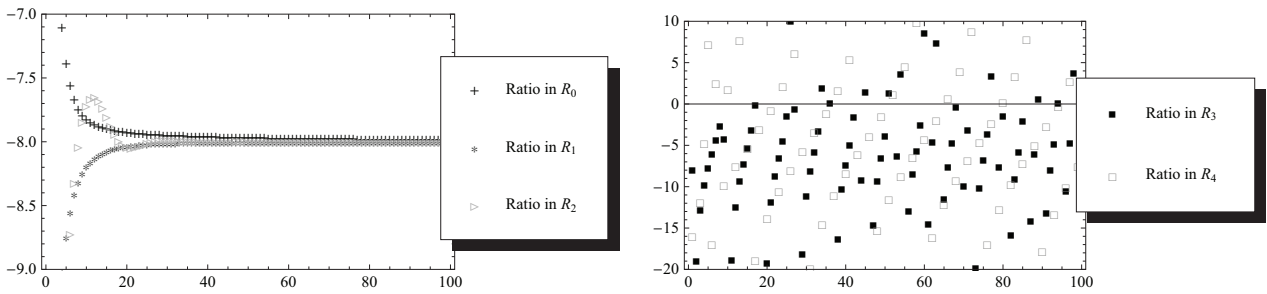


FIG. 1: Plot of ratio of the coefficients of series expansion of $\mathcal{R}_\ell = \chi^{(\ell+1)}/\chi^{(\ell)}$ ($\ell = 0, 1, 2, 3, 4$). The series of \mathcal{R}_ℓ for $\ell = 0, 1, 2$ show alternative sign. We have confirmed the alternative property up to 100 terms. All three series exhibit convergence to -8 , which signals the radius of convergence $1/8 = 0.125$. On the other hand, in \mathcal{R}_3 and \mathcal{R}_4 , the coefficients do not show alternative property and the ratio fluctuate with large amplitude, thus exhibit no sign of convergence. By these results, we expect that the series-property changes drastically at the jump from $\ell = 2$ to $\ell = 3$. Note that these results are obtained solely from the high temperature expansion.

B. Delta expansion

Since the details of the δ -expansion is discussed and explained in the past literatures, we do not repeat them. Rather, we just state the essential idea and results and show the outline of the usage.

The δ -expansion is related to the dilation around the critical point $M = 0$. The dilation needs precise information of β_c if τ is employed as the basic parameter. However, in the study under the second order transition, the critical point is given in the mass as $M = 0$ in the manner independent of the models. Hence, the dilation is simply implemented when the basic parameter is chosen as the mass. We make dilation in the thermodynamic function by the replacement of M by $(1 - \delta)/t$ where t^{-1} stands for the rescaled mass parameter. After the expansion of the function in δ up to a relevant order, setting $\delta = 1$ gives a non-trivial dilated function in terms of t . Due to the breaking of the regular correspondence between M and t at $\delta = 1$, the physical interpretation of t in this limit is obscured. However, we are able to confirm that the limit $t \rightarrow \infty$ in the resultant function exactly recovers the correct limit of the original function (see for details [1, 7]). Denoting the δ -expansion by the operation symbol D , the important result of δ -expansion is summarized by

$$D[M^{-n}] = C_{N,n} t^n, \quad n = 0, 1, 2, 3, \dots \quad (24)$$

where $C_{N,n}$ denotes the binomial coefficient

$$C_{N,n} = \frac{N!}{n!(N-n)!}. \quad (25)$$

Here we note that N stands for the order of the full expansion. In this sense, the δ -expansion depends on the order of the large mass expansion. If one faces with the comparison of the δ -expanded t series with the series valid near the

critical point where $t \gg 1$, it is empirically known that the good matching occurs for the prescription

$$D[M^p] = C_{N,-p} t^{-p} = \frac{\Gamma(N+1)}{\Gamma(-p+1)\Gamma(N+p+1)} t^{-p}. \quad (26)$$

From above we find for positive integer p that $D[M^p] = 0$. This ensures the considerable suppression of the regular contribution involved in the expansion (14).

Thus, the δ -expansion on the large mass expansion to N th order,

$$\mathcal{R}_\ell = a_0^{(\ell)} + \sum_{n=1}^N a_n^{(\ell)} x^n \quad (27)$$

supplies the following series in t ,

$$D[\mathcal{R}_\ell] = D\left[a_0^{(\ell)} + \sum_{n=1}^N a_n^{(\ell)} x^n\right] = a_0^{(\ell)} + \sum_{n=1}^N C_{N,n} a_n^{(\ell)} t^n := \bar{\mathcal{R}}_\ell. \quad (28)$$

Note that $a_0^{(\ell)} = 0$ for $\ell = 0$ and $a_0^{(\ell)} = 1$ for $\ell \geq 1$. As numerical check, we have drawn the behaviors of $\bar{\mathcal{R}}_\ell$. See Fig. 2, which shows the plots of $\bar{\mathcal{R}}_\ell$ at $\ell = 0, 1, 2, 3$. Except for $\ell = 3$, the δ -expanded functions have effective regions roughly twice wider compared to those in the original series. And we remark that the approach to the limit $q_1 = 7/8$ is convincing, which is the evidence that the δ -expanded series contains within its effective region the scaling region (In the present case, the scaling region is the plateau with wide range). It seems that $\bar{\mathcal{R}}_2$ gives best realization of q_1 ,

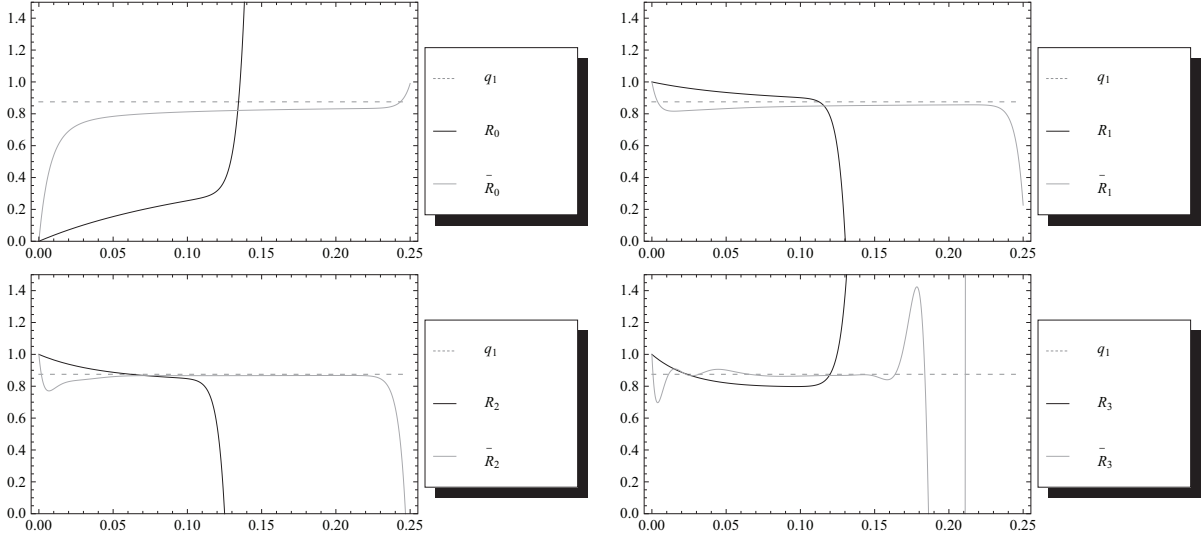


FIG. 2: The plot of $\mathcal{R}_\ell = \chi^{(\ell+1)}/\chi^{(\ell)}$ and $\bar{\mathcal{R}}_\ell = D[\mathcal{R}_\ell]$ ($\ell = 0, 1, 2, 3$) at 25th order in the large ℓ expansion. The horizontal axis stands for x for \mathcal{R}_ℓ and t for $\bar{\mathcal{R}}_\ell$.

though the reason why $\ell = 2$ provides best behavior is not known to us. The behavior of $\bar{\mathcal{R}}_3$ is oscillatory and not suitable for our purpose. This is true also for $\bar{\mathcal{R}}_4$. The first three plots share a common property that all of them has alternative coefficients (see also Tables V–IX in Appendix).

III. ESTIMATING γ AND SUB-LEADING EXPONENT

A. Non-parametric case

In the previous section, we confirmed that the δ -expansion revealed the behavior in the vicinity of the critical point $t = \infty$ at the region where t is small. This is the effect of dilation around the critical point. Having prepared the estimation environment, let us set up the reference of the estimation by adopting naive use of the principle of minimum sensitivity (PMS) [8].

First of all, we regard the plateau as the realization of the scaling region. This identification is natural because it is quite conceivable that the asymptotic scaling behavior is just the convergence to the limit q_1 and stationary. Actually, watching plot of $\bar{\mathcal{R}}_0$ (see Fig. 2), we see that the point of least variation would approximate the limit q_1 . Also for $\bar{\mathcal{R}}_1$ and $\bar{\mathcal{R}}_2$, the stationary points provide the approximation of q_1 due to the same reason. In this manner, we can estimate q_1 . The result is summarized in Table I.

TABLE I: Estimation result of $q_1 = 7/8 = 0.875$.

$\bar{\mathcal{R}}_\ell$	15	20	25	30	35	40	45	50
$\bar{\mathcal{R}}_0$	0.8169773	0.8276445	0.8317319	0.8370296	0.8391402	0.8424476	0.8437536	0.8460722
$\bar{\mathcal{R}}_1$	0.8497474	0.8530335	0.8563755	0.8577449	0.8596025	0.8603685	0.8615928	0.8620879
$\bar{\mathcal{R}}_2$	0.8661683	0.8662792	0.8675665	0.8679249	0.8686629	0.8689309	0.8694071	0.8695930

As indicated in the plots in Fig. 2, the best result comes from $\bar{\mathcal{R}}_2$. However, the accuracy is not satisfactory yet. At 25th order as a reference result, the error is about 1 percent. For the accurate estimation of the exponent q_1 , some additional device is needed to reduce the correction to the asymptotic scaling. The device we employ here is the parametric extension of thermodynamic functions proposed in [1].

B. Parametric extension

In the case of \mathcal{R}_ℓ , the corresponding parametric extension gives

$$\psi_\ell(\alpha_1, \alpha_2, \dots; x) = \left\{ 1 + \alpha_1 x \frac{d}{dx} + \alpha_2 \left(x \frac{d}{dx} \right)^2 + \dots \right\} \mathcal{R}_\ell. \quad (29)$$

We note that the differentiation deletes the leading constant q_1 (see (18)). Hence, irrespective of the values of α_k , ψ_ℓ converges to $q_1 = \gamma/2\nu$ as is easily understood from (18). In the limit of $x \rightarrow \infty$, the independence of $\lim_{x \rightarrow \infty} \psi_\ell$ over α_k would be apparent, but in situation where the limit cannot be taken, the appropriate value of the parameters in the estimation work would exist. Since $\psi_\ell \sim q_1 + C_2/C_1(q_2/q_1)^\ell(q_2 - q_1)(1 + (q_2 - q_1)\alpha_1 + (q_2 - q_1)^2\alpha_2 + \dots)x^{q_2 - q_1} + \dots$, the choice of parameters satisfying

$$1 + (q_2 - q_1)\alpha_1 + (q_2 - q_1)^2\alpha_2 + \dots = 0 \quad (30)$$

makes the leading correction vanishing. If the K parameters are introduced, it may be possible to delete or reduce considerably the first K corrections. In this work, however, we confine ourselves with the extension of just single parameter. Then, it is apparent that the reduction of the leading correction needs the value of $q_2 - q_1$, since (30) reads in this case

$$1 + (q_2 - q_1)\alpha_1 = 0 \quad (31)$$

which yields $\alpha_1 = -(q_2 - q_1)^{-1}$. Of course, we must work in the situation where the value of $q_2 - q_1$ is not known to us. Being blind on $q_2 - q_1$, we must specify optimal value of α_1 within the truncated large mass series. For the task, it is crucial to consult with the δ -expanded series where the scaling region is observed in the small t expansion. The effective reduction of the leading order correction would make plateau flatter than the parameter-less original function. Thus, we extend the principle of minimum sensitivity (PMS) to fix optimal α_1 . The step goes as follows: First, we take that optimal value of α_1 is given by the case where the stationary point of $\bar{\psi}_\ell$ becomes "maximally stationary". The maximal stationarity means that, at the stationary point, following simultaneous conditions should hold,

$$\bar{\psi}_\ell^{(1)} = \bar{\mathcal{R}}_\ell^{(1)} + \alpha_1 \bar{\mathcal{R}}_\ell^{(2)} = 0, \quad (32)$$

$$\bar{\psi}_\ell^{(2)} = \bar{\mathcal{R}}_\ell^{(2)} + \alpha_1 \bar{\mathcal{R}}_\ell^{(3)} = 0. \quad (33)$$

By imposing above conditions on the small t series of $\bar{\psi}_\ell$, we may obtain optimal $\alpha_1 = \alpha_1^*$ and the point $t = t^*$ at which q_1 should be estimated by $\bar{\psi}_\ell(\alpha_1^*, t^*)$. However, in some cases, there exists no solution within plateau for the second condition (33), though the first condition (32) always has solution. In this case, we instead take a loose

condition which requires that the absolute value of the second derivative, $|\bar{\psi}_\ell^{(2)}|$, is minimum at the point where $\bar{\psi}_\ell$ is stationary.

Applying such a generalized PMS condition, we find sets of solutions (α_1^*, t^*) . In general the set is not unique at a given order. Among them most reliable set would be the one with largest $t^* = t_{best}$. The effectivity of this prescription manifests themselves by the fact that, at large orders, t_{best} signals the limit at large- t side of the plateau. Thus, we obtain the best estimations by

$$q_1 \sim \bar{\psi}_\ell(\alpha_1^*, t_{best}), \tag{34}$$

$$(q_1 - q_2)^{-1} \sim \alpha_1^*. \tag{35}$$

The results are shown in Figs. 3, 4 and Tables II and III.

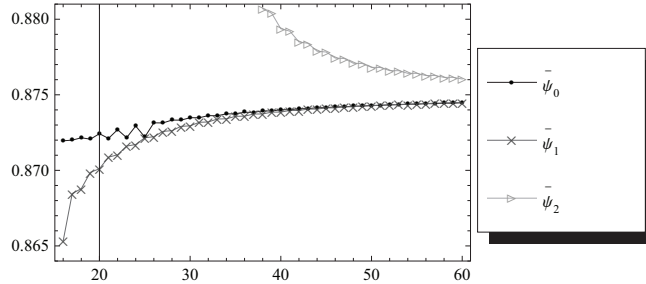


FIG. 3: The one-parameter estimation of $q_1 = 7/8$ with $\bar{\psi}_\ell$ ($\ell = 0, 1, 2$).

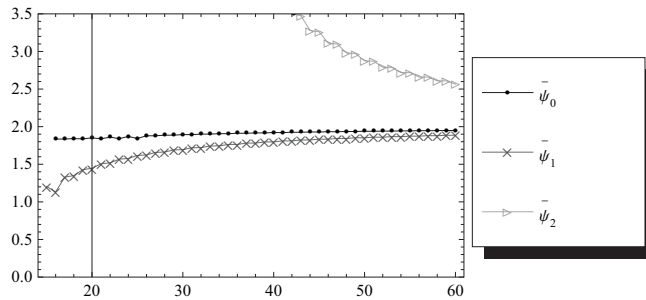


FIG. 4: The one-parameter estimation of $(q_1 - q_2)^{-1} = 2$ with $\bar{\psi}_\ell$ ($\ell = 0, 1, 2$).

TABLE II: Estimation result of $q_1 = 7/8 = 0.875$. The case of $\ell = 2$ has poor realization of the scaling behavior and estimation. This is the reason of the blanks in the table.

$\bar{\psi}_\ell$	20	25	30	35	40	45	50
$\bar{\psi}_0$	0.8724245	0.8722058	0.8734845	0.8737348	0.8740136	0.8741421	0.8743013
$\bar{\psi}_1$	0.8701054	0.8721568	0.8729444	0.8735773	0.8738560	0.8741360	0.8742653
$\bar{\psi}_2$					0.8793704	0.8778297	0.8768158

By the numerical experiment, we find that, for q_1 , the sequences exhibit clear convergence to the correct limit in $\bar{\psi}_0$ and $\bar{\psi}_1$. For instance, $\bar{\psi}_0$ gives estimation $q_1 \sim 0.8729$ (error ~ 0.24 percents) and 0.8722 (error ~ 0.32 percents) at $N = 24$ and 25 , respectively. $\bar{\psi}_2$ implies slow convergence to the correct limit. In the same manner with the q_1 -sequence, the sequence of optimal α_1 in $\bar{\psi}_0$ strongly indicates the correct value of $(q_1 - q_2)^{-1} = 2$. The cases $\ell = 0, 1$ provide similar results both of those are satisfactory. On the other hand, the case $\ell = 2$ ended in poor results.

TABLE III: Estimation result of $(q_1 - q_2)^{-1} = 2$ via optimized α_1 . The case of $\ell = 2$ has poor realization of the scaling behavior and estimation. This is the reason of the blanks in the table.

$\bar{\psi}_\ell$	20	25	30	35	40	45	50
$\bar{\psi}_0$	1.8505000	1.8425165	1.8944572	1.9067215	1.9216758	1.9291058	1.9389608
$\bar{\psi}_1$	1.4461756	1.6177268	1.6943348	1.7636010	1.7971446	1.8337286	1.8518406
$\bar{\psi}_2$					3.8234502	3.2604537	2.8749102

IV. CONCLUDING REMARKS

To conclude our investigation, the δ -expansion in the non-parametric scheme reveals that the cases $\ell = 0, 1, 2$ manifest themselves that the scaling region emerges in the small t region and the correct value of q_1 is indicated. In the parametric cases with single parameter, the accuracy of q_1 and $(q_1 - q_2)^{-1}$ estimation is highly improved. However, the case $\ell = 2$ has failed in improving the accuracy. Let us consider the reason of failure for the $\ell = 2$ case.

The point is that, in the parametric scheme, the derivatives of \mathcal{R}_ℓ enters into the job. For instance, we find from (34) and (35), the derivatives to the third one are needed to achieve the estimation procedures. Then, for the success of the procedures, derivatives to the third one must show the approximate scaling around the estimation region of t (plateau region of $\bar{\mathcal{R}}_\ell$). So let us focus on the scaling behaviors of derivatives. From (18), it follows that

$$\bar{\mathcal{R}}_\ell^{(k)} = (C_2/C_1)C_{N, q_2 - q_1}(q_2/q_1)^\ell (q_2 - q_1)^k t^{q_2 - q_1} + \dots \quad (36)$$

To begin with we remind that for all ℓ ,

$$\lim_{t \rightarrow \infty} \bar{\mathcal{R}}_\ell^{(k)} = 0. \quad (37)$$

Since $q_2 - q_1 < 0$, the coefficient changes sign under the differentiation. Then, both of $\bar{\mathcal{R}}_0^{(1)}$ and $\bar{\mathcal{R}}_1^{(1)}$ tend to zero from above and q_2 is found to be positive. Hence, the sign of the front factor does not change with ℓ . As is understood from the plots of derivatives (see Fig. 4), in the cases $\ell = 0, 1$, those derivatives may be said as realizing the scaling. However, in the case $\ell = 2$, the derivatives do not show the scaling behavior yet. In fact, scaling behavior of $\mathcal{R}_2^{(1)}$ is not clearly seen even at 100th order (We note that these features can be observed solely from δ -expanded small t series.). This is the reason why parametric extension of \mathcal{R}_2 does not bring about improvement.

As a final remark, we briefly compare our results in parametric extension with the traditional approaches. Representative technique among them is the so-called Padé approximant method of first derivative of $\log \chi$ in high temperature series (series in β) [3]. The approach provides accurate β_c (for instance $\beta_c \sim 0.4406838 \dots$ at 20th order in the diagonal approximant) and, using the result, gives $\gamma \sim 1.7496 \dots$. The accuracy is quite high. Thus, our approach is not good in the accuracy of estimation. There is one advantage in our approach, however. In the traditional approaches such as the representative one has no unique estimation at a given order. For example, there are other values of estimation depending on the choice of degrees of denominator and numerator of Padé approximants. On the other hand, we can identify which is the best one among a few candidates in our approach. This selection becomes possible since the approximate critical behaviors of \mathcal{R}_ℓ becomes visible under the δ -expansion.

There are related subjects not discussed in this work: The parametric extension of the original thermodynamic quantities is not limited to the single parameter case. Two- and three- parameter extension is a natural next step in our approach. The study along this road is now under the progress. We have found under yet rough examination that the accuracy of estimation is further improved, but, unfortunately, the clear scaling begins to show at large order. The method presented in this work is applicable also at low temperature. It is also interesting to apply the present approach to other thermodynamic functions as the specific heat, the magnetization and the amplitude ratios and so on. As another subject, the choice of the basic mass variable should also be studied. Two candidate of the mass, second moment mass and the diagonal exponential mass, would be compared with each other on the results they would supply. After the completion of these subjects, we like to report results in the subsequent publications.

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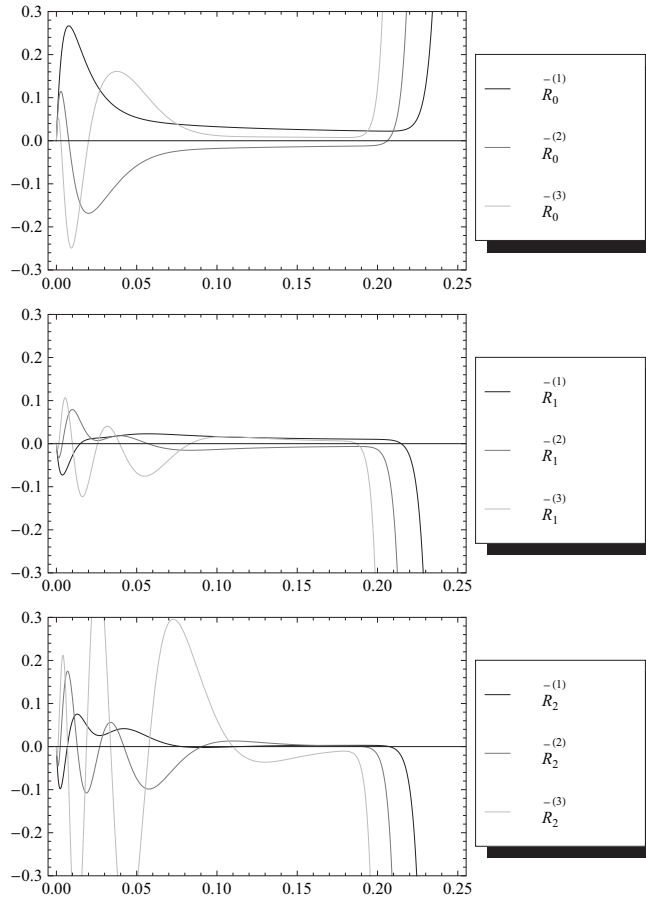


FIG. 5: $\bar{\mathcal{R}}_\ell^{(k)}$ for $\ell = 0, 1, 2$ and $k = 0, 1, 2, 3$ at 25th order. For $\ell = 0$ and 1, the scaling region of derivatives are well developed. On the contrary, for $\ell = 2$, the scaling region is not developed yet.

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Appendix

TABLE IV: Coefficients of $\chi(x) = 1 + \sum_{n=1} a_n x^n$ in the series expansion at $x \ll 1$ ($M \gg 1$).

n	a_n	n	a_n
1	4	21	21275386763804768
2	-4	22	-160006809343054864
3	16	23	1206597595055984816
4	-84	24	-9121198369179912432
5	496	25	69106863875292721536
6	-3120	26	-524680254679683977840
7	20416	27	3991217801247779845008
8	-137300	28	-30415396853284535164192
9	942368	29	232169581608188044281504
10	-6571808	30	-17749770132732227415243684
11	46422672	31	13589773757322695502106928
12	-331425504	32	-104189603870648624732261332
13	2387361104	33	799822083737000041347307488
14	-17328288880	34	-6147330757104867389693195232
15	126603329808	35	47301411121280638308539291728
16	-930294191876	36	-364358216053104960233960685968
17	6870391514160	37	2809472823077732343284217635680
18	-50965973697504	38	-21684016505427268568571357447824
19	379584845946000	39	167514287529323896647618797568880
20	-2837208508428432	40	-1295213666986423469602452039905120

TABLE V: Coefficients of $\chi^{(1)}/\chi^{(0)} = \sum_{n=1} a_n^{(0)} x^n$ in the series expansion at $x \ll 1$ ($M \gg 1$).

n	$a_n^{(0)}$	n	$a_n^{(0)}$
1	4	21	1702823724672856736
2	-24	22	-13501596184503027680
3	160	23	107091803029662561424
4	-1136	24	-849705201298923243520
5	8384	25	6743862211146068408384
6	-63360	26	-53538558608960941163360
7	485888	27	425140903442007642553648
8	-3760864	28	-3376752124592333432399232
9	29288176	29	26826151806712841937551776
10	-229044704	30	-213159226806584799070943840
11	1796618608	31	1694067888051261417465988304
12	-14124299840	32	-13465836756628212002914677632
13	111232195728	33	107055058630461662398852456816
14	-877185399072	34	-851232593675722299370220140384
15	6925285123760	35	6769440346033398167021761676528
16	-54725053955264	36	-53841525398447790365876496293504
17	432786546475136	37	428290955376655997494957586184224
18	-3424925637964512	38	-3407332790404117881313638891492448
19	27119362491604272	39	27110758945124257909506817017689232
20	-214846102451691136	40	-215733626693465249999762284727613824

TABLE VI: Coefficients of $\chi^{(2)}/\chi^{(1)} = 1 + \sum_{n=1} a_n^{(1)} x^n$ in the series expansion at $x \ll 1$ ($M \gg 1$).

n	$a_n^{(1)}$	n	$a_n^{(1)}$
1	-2	21	-5681180325529261272
2	20	22	45585039485774965592
3	-188	23	-365622948610387580424
4	1696	24	2931554994773309328576
5	-14832	25	-23498378896007252038632
6	126800	26	188309058357448190591800
7	-1066536	27	-1508735921272122489591320
8	8866944	28	12085838557513413548876288
9	-73099520	29	-96799409028653454848531512
10	598917240	30	775193516919679155773697240
11	-4884337240	31	-6207215526175544707533472648
12	39692344624	32	49698053761202350389581215232
13	-321670811568	33	-397872043090052734523304056656
14	2601181273480	34	3185031697906826149624124991320
15	-20997665014008	35	-2549496998601597957766654717096
16	169260623296960	36	204065318897860700935135362721984
17	-1362814550375976	37	-1633280963799008226225446117860920
18	10962294305104952	38	13071704883034769184711013037937368
19	-88109033749900600	39	-104612959133323675229583441074445768
20	707702384740998656	40	837187318006362302642544817683192864

TABLE VII: Coefficients of $\chi^{(3)}/\chi^{(2)} = 1 + \sum_{n=1} a_n^{(2)} x^n$ in the series expansion at $x \ll 1$ ($M \gg 1$).

n	$a_n^{(2)}$	n	$a_n^{(2)}$
1	-4	21	-15040730135129492272
2	56	22	121004881555705996872
3	-640	23	-972426936117955593200
4	6480	24	7803749063758302892096
5	-60224	25	-62536717228676702935024
6	525440	26	500525348476468610236296
7	-4373632	27	-4002161071827691228836880
8	35166208	28	31979699003934529220811968
9	-275926528	29	-255442783197624248573622832
10	2131274216	30	2040145044108510196576411400
11	-16329778128	31	-16295163453848209150086894384
12	124926689328	32	130178881834711284732453163008
13	-959233922144	33	-1040241930805335683349755457024
14	7419013365176	34	8314703351472082671211269216968
15	-57904294231760	35	-66476616064355872759999824896592
16	456132757502656	36	531596887846798342490795731022592
17	-3622486505331568	37	-4251720114986581077363369713314544
18	28950109717348424	38	34008910282093517770619019429908936
19	-232333096023161872	39	-272048440397204492233039080683024624
20	1868720741877011520	40	2176251364472264001414753196521168288

TABLE VIII: Coefficients of $\chi^{(4)}/\chi^{(3)} = 1 + \sum_{n=1} a_n^{(3)} x^n$ in the series expansion at $x \ll 1$ ($M \gg 1$).

n	$a_n^{(3)}$	n	$a_n^{(3)}$
1	-8	21	-47163020009908433952
2	152	22	412116896023797485816
3	-1952	23	-2699305167935251780896
4	19216	24	11968257756316458709056
5	-149568	25	-17609179361962050886368
6	904832	26	-176308656907167515433512
7	-3932160	27	104349065828589559785376
8	10355712	28	3621304717086125252834028
9	-44546048	29	-656440945752077901138957472
10	1292483832	30	7280425095959386195204357752
11	-24361161184	31	-58726611083812091179172774560
12	304417044592	32	341020340137399083179694436352
13	-2844873583872	33	-1119767807866941643624038227584
14	20495746694344	34	-2176067142302729239126317062056
15	-109034249191392	35	46891539299459592211807394564000
16	349883316909760	36	3292725713811400499298226387072
17	-50163076414560	37	-6615630180814263178318159903485600
18	1424993451633368	38	108447822923488109588859786319705592
19	-206345171917851232	39	-1109639966062292122355722655120070432
20	3968211883960147136	40	8153205060859798662287075542995817632

TABLE IX: Coefficients of $\chi^{(5)}/\chi^{(4)} = 1 + \sum_{n=1} a_n^{(4)} x^n$ in the series expansion at $x \ll 1$ ($M \gg 1$).

n	$a_n^{(4)}$	n	$a_n^{(4)}$
1	-16	21	-47163020009908433952
2	392	22	412116896023797485816
3	-4672	23	-2699305167935251780896
4	22224	24	11968257756316458709056
5	162304	25	-17609179361962050886368
6	-2746240	26	-176308656907167515433512
7	-3932160	27	104349065828589559785376
8	10355712	28	3621304717086125252834028
9	-44546048	29	-656440945752077901138957472
10	1292483832	30	7280425095959386195204357752
11	-24361161184	31	-58726611083812091179172774560
12	304417044592	32	341020340137399083179694436352
13	-2844873583872	33	-1119767807866941643624038227584
14	20495746694344	34	-2176067142302729239126317062056
15	-109034249191392	35	46891539299459592211807394564000
16	349883316909760	36	3292725713811400499298226387072
17	-50163076414560	37	-6615630180814263178318159903485600
18	1424993451633368	38	108447822923488109588859786319705592
19	-206345171917851232	39	-1109639966062292122355722655120070432
20	3968211883960147136	40	8153205060859798662287075542995817632